Fine and Hyperfine Splittings of Charmonium and Bottomonium: An Improved Perturbative QCD Approach
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We extend the perturbative QCD-based formalism that was developed in our previous work, and compute the fine and hyperfine splittings of the bottomonium and charmonium spectra. All the corrections up to $O(q^2)$ are included in the computations. We find agreement (with respect to theoretical uncertainties) with the experimental values whenever available and give predictions for not yet observed splittings. We show that the QCD potential obtained with our scale fixing procedure is consistent with lattice calculations.

For decades, phenomenological potential models have been used to calculate quarkonium spectra. Problem: Renormalons! Illustration: large $\beta_0$ approximation.

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  \text{Physical quantity: Energy of quark pair} \quad E_{\text{q-q}}(r) = 2m_{\text{q}} + V_{\text{QCD}}(r),

  \text{with} \quad m_{\text{q}} = \frac{1}{2}(1 + \frac{\alpha(\mu)}{\alpha(\m))}) ( \frac{d_0}{4} + \frac{\alpha(\mu)}{\alpha(\m))} d_2 ),

  V_{\text{QCD}}(r) = -\frac{4\times\alpha(\mu)}{r} + \frac{\alpha(\mu)}{\alpha(\m))} d_2.
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Achieving decoupling of IR degrees of freedom (renormalon cancellation) at each order of the perturbative expansion by re-expressing the quark pole mass in terms of a short-distance mass, such as the MS mass, and expanding $m_{\text{q}}$ and $V_{\text{QCD}}(r)$ in the same coupling constant.

Two scale fixing prescriptions:
1. $\mu = \mu(r)$ fixed by demanding stability against scale variation:
   \[ \mu(r) \frac{dE_\mu(r,\alpha_0,\alpha_0(r))}{d\mu} |_{\mu=\mu(r)} = 0.\]
2. $\mu = \mu(r)$ fixed to minimum of absolute value of known term $/O(q^2)$ of $E_{\text{q-q}}(r)$:
   \[ \mu(r) \frac{dE_\mu(r,\alpha_0,\alpha_0(r))}{d\mu} |_{\mu=\mu(r)} = 0.\]

Perturbative series of quarkonium organised as: $H_\text{QCD} = H_\text{MS} + E_\text{QCD}(r)$
Unconventional, because $H_\text{QCD}$ includes $O(1/r^2)$ and $O(1/r^3)$ terms from QCD potential $V_{\text{QCD}}(r)$.

Slightly improved potential $E_{\text{improved}}(r)$:
1. Intermediate distances, $r_{\text{coul}} < r < r_{\text{QCD}}$:
2. Short distances, $r < r_{\text{coul}}$:
3. Long distances, $r > r_{\text{QCD}}$:

Formally, using (analytic) solutions to the Coulomb potential or using numeric solutions to the QCD-potential is equivalent within the theoretical uncertainty. Using the agreement with experimental spectra, however, can be greatly improved by using the latter! Wave functions differ very much: QCD pot. rises linearly; sugar! Fine Splitting $\not\rightarrow 1/r^3$ !

Find energy levels by numerically solving: $E^{\text{improved}}(r) = \text{Min} \{ U + W_A + W_{0A} \} (\psi)$

Also use $O(1/r^2)$ operators to reduce scale dependence!

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  \begin{align*}
  E_{\text{improved}}(r) &= \text{Min} \{ U + W_A + W_{0A} \} (\psi) \\
  \text{Convergence of } E_{\text{improved}} \text{ for } r > 2m_q &\approx 5 \text{ GeV}^{-1}. \quad \text{All numbers in GeV}.
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Caution: Comparing perturbative QCD to quenched lattice calculations, therefore no direct connection to real world. Use Sommer scale:

Expoential fit: $d_{QCD}(r) = 1.65, \quad m_{\pi} = 0.602 r_{QCD}^{-1}$ (ALPHA coll.), $\rho_{QCD} = 400$ MeV ($\rho_{QCD} \approx 0.5f_{QCD}$).

Stability criterion: Energies determined with scales fixed by $\rho_{QCD}$.

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  \begin{align*}
  \text{Comparison of our potential with lattice calculations} \\
  \text{Lattice most reliable in quenched approximation, therefore compare quenched lattice to QCD with } r_{\text{QCD}} = 0 \not\rightarrow \text{scaleless}.
  \end{align*}
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  \begin{align*}
  \text{Comparisons pert. QCD } \not\rightarrow \text{lattice QCD have been made for some time, but with fixed } r_{QCD} \text{ and with only up to } r_{QCD} = 0 \text{ (r_{QCD} = 400 GeV, i.e. } r_{QCD} \approx 0.5f_{QCD}).
  \end{align*}
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  \begin{align*}
  \text{With the minimum sensitivity scale fixing procedure, the perturbative series remains stable for large values of } r_{QCD}.
  \end{align*}
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Conclusions:
Minimum sensitivity scale fixing procedure allows stable determination of QCD potential at rather large distances, in agreement with lattice calculations.

This potential can be used to calculate spectra and especially level splittings in heavy quarkonia.

Not completely pure QCD scale fixing procedure, ... but no model parameters!

Only input is $m_{\text{q}}$, $m_{\text{r}}$, $\alpha_s(MZ)$. 

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