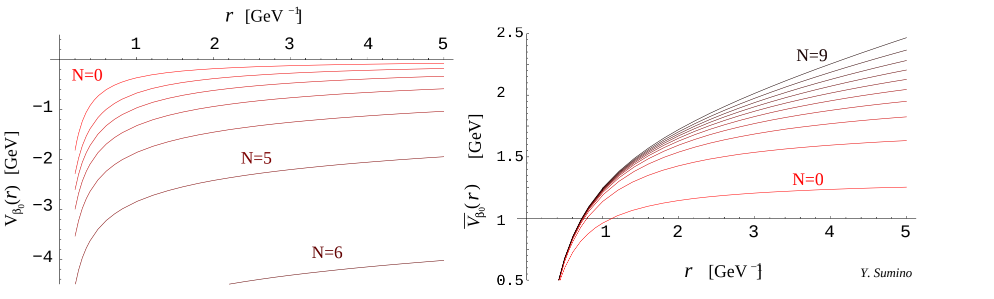
Fine and Hyperfine Splittings of Charmonium and Bottomonium: An Improved Perturbative QCD Approach

Stefan Recksiegel (Technische Universität München) and Yukinari Sumino (Tohoku University)

We extend the perturbative QCD–based formalism that was developed in our previous work, and compute the fine and hyperfine splittings of the bottomonium and charmonium spectra. All the corrections up to $\mathcal{O}(\alpha_S^5 m)$ are included in the computations. We find agreement (with respect to theoretical uncertainties) with the experimental values whenever available and give predictions for not yet observed splittings. We show that the QCD potential obtained with our scale fixing procedure is consistent with lattice calculations.

For decades, phenomenological potential models have been used to calculate quarkonium spectra. Problem: Renormalons ! Illustration: large β_0 approximation.



Physical quantity: Energy of $q\bar{q}$ pair ! $\quad E_{\rm tot}(r) = 2m_{\rm pole} + V_{\rm QCD}(r)$,

$$\text{with} \qquad m_{\rm pole} = \overline{m}\left(1 + \frac{4}{3}\frac{\alpha_S(\overline{m})}{\pi} + \left(\frac{\alpha_S(\overline{m})}{\pi}\right)^2 d_1 + \left(\frac{\alpha_S(\overline{m})}{\pi}\right)^3 d_2\right),$$

$$V_{\rm QCD}(r) = -\frac{4}{3}\frac{\alpha_S(\mu)}{r}\left[1 + \frac{\alpha_S(\mu)}{4\pi}(2\beta_0\ell + a_1) + \frac{\alpha_S(\mu)^2}{(4\pi)^2}\left\{\beta_0^2\left(4\ell^2 + \frac{\pi^2}{3}\right) + 2(\beta_1 + 2\beta_0 a_1)\ell + a_2\right\}\right]$$

Achieve decoupling of IR degrees of freedom (renormalon cancellation) at each order of the perturbative expansion by re–expressing the quark pole mass in terms of a short–distance mass, such as the $\overline{\rm MS}$ mass, and expanding $m_{\rm pole}$ and $V_{\rm QCD}(r)$ in the same coupling constant.

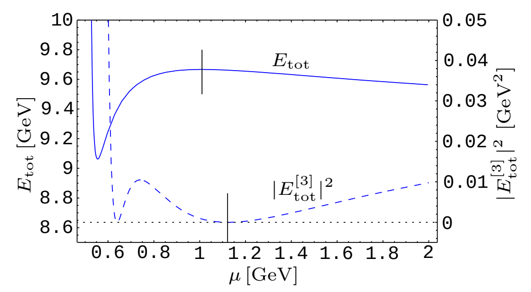
Two scale fixing prescriptions:

1. $\mu = \mu_1(r)$ fixed by demanding stability against scale variation:

$$\mu\frac{d}{d\mu}E_{\rm tot}(r;\overline{m},\alpha_S(\mu))\,\big|_{\mu=\mu_1(r)} = 0$$

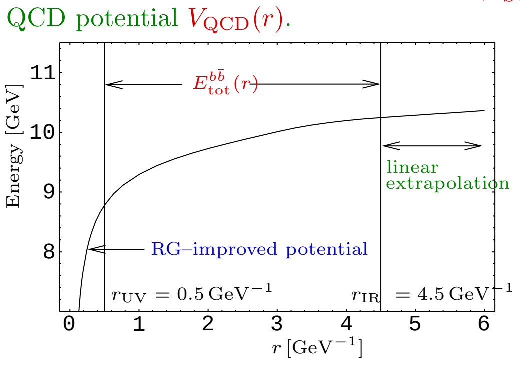
2. $\mu = \mu_2(r)$ fixed to minimum of absolute value of last known term $[\mathcal{O}(\alpha_S^3)]$ of $E_{\rm tot}(r)$:

$$\mu\frac{d}{d\mu}\left(E^{(3)}_{\rm tot}(r;\overline{m},\alpha_S(\mu))\right)^2\,\big|_{\mu=\mu_2(r)} = 0$$



Perturbative series of quarkonium organised as: $H_0 = 2m_Q + \frac{\vec{p}^{\,2}}{m_Q} + V_{\rm QCD}(r)$

Unconventional, because H_0 includes $\mathcal{O}(\alpha_S^2) = \mathcal{O}(1/c)$ and $\mathcal{O}(\alpha_S^3) = \mathcal{O}(1/c^2)$ terms from QCD potential $V_{\rm QCD}(r)$.



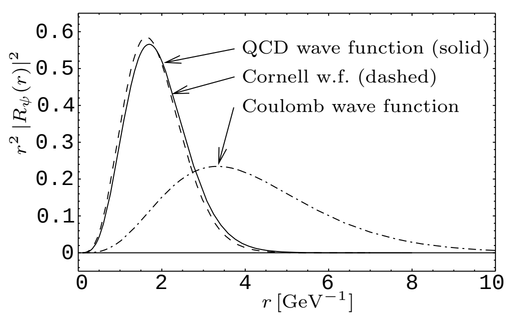
Slightly improved potential $E_{\rm imp}(r)$:
1. Intermediate distances, $r_{\rm UV} < r < r_{\rm IR}$: $E_{\rm imp}(r) = E^{b\bar{b}}_{\rm tot}(r) = 2m_b + V_{\rm QCD}(r)$
2. Short distances, $r < r_{\rm UV}$: renormalization–group improved QCD potential
3. Long distances, $r > r_{\rm IR}$: Linear extrapolation

Find energy levels by numerically solving: $H_0^{\rm (imp)}\,|\,\psi\,\rangle = E^{(0)}_\psi\,|\,\psi\,\rangle$, $\quad H_0^{\rm (imp)} = \frac{\vec{p}^{\,2}}{m_Q} + E_{\rm imp}(r)$

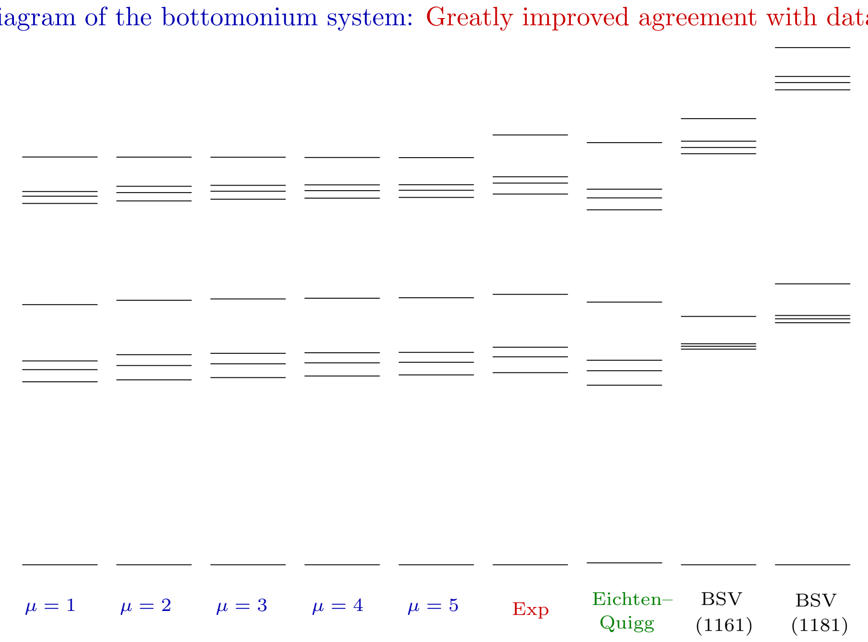
The other terms are treated as perturbations: $\Delta E_\psi = \langle\,\psi\,|\,(U + W_A + W_{NA})\,|\,\psi\,\rangle$

Also use $\mathcal{O}(1/c^3)$ operators to reduce scale dependence !

Formally, using (analytic) solutions to the Coulomb potential or using numeric solutions to the QCD–potential is equivalent within the theoretical uncertainty. The agreement with experimental spectrum, however, can be greatly improved by using the latter ! Wave functions differ very much: QCD pot. raises linearly: squeezes WF ! Fine Splitting $\sim 1/r^3 \Longrightarrow$ Big difference !



Level Diagram of the bottomonium system: Greatly improved agreement with data !



Exp = Experimental data, Eichten–Quigg = potential model,
BSV = QCD calculation with $H_0 = p^2/m + V_{\rm Coulomb}$, $\alpha_s(M_Z) = \{0.1161, 0.1181\}$
Agreement with experiment of our results is comparable to that of model calculations, but:

Model calculations: Functional form of potential is put in by hand !
Our formalism: Scale fixing procedure and Organisation of the perturbative series.
Input: Only ($\overline{MS}$–) masses and $\alpha_s(M_Z)$!

We want to make predictions for the level splittings in the bottomonium system, but the formalism even works for charmonium ! All values in MeV

Level splitting	Exp.	Potential model				Lattice		Pert. QCD	
		EQ	MZ	EFG	HFM	CP	LM	PT	RS
$\chi_{c1}(1P) - \chi_{c0}(1P)$	95	50	81	86	72	79	–	–	55
$\chi_{c2}(1P) - \chi_{c1}(1P)$	46	21	50	46	49	35	–	–	42
$J/\Psi - \eta_c(1S)$	117	117	117	117	117	85	–	–	88
$\Psi(2S) - \eta_c(2S)$	92/32	78	72	98	92	43	–	–	37
$\chi_c^{\rm cog}(1P) - h_c(1P)$	−0.9	0	0	0	9	1.5	–	−1.4	−0.7
$\Upsilon(1S) - \eta_b(1S)$	(160)	87	57	60	45	–	51	–	44
$\Upsilon(2S) - \eta_b(2S)$	–	44	28	30	28	–	–	–	21
$\Upsilon(3S) - \eta_b(3S)$	–	41	20	27	23	–	–	–	11
$\chi_b^{\rm cog}(1P) - h_b(1P)$	–	0	0	−1	1	–	–	−0.5	−0.4
$\chi_b^{\rm cog}(2P) - h_b(2P)$	–	0	0	−1	0	–	–	−0.4	−0.2

(EQ = Eichten/Quigg 94: Simple model; MZ = Motyka/Zalewski 98: More sophisticated model; EFG = Ebert/Faustov/Galkin 03; HFM = Haysak/Fekete/Morokhovych/... 03; CP = CP-PACS coll. 02; LM = Liao/Manke 02; PT = Pantaleone/Tye 98; RS = this work)

Even for charmonium the formalism works well ! Should be reliable for bottomonium.

With a (phenomenologically motivated) non–standard organisation of the perturbative series, it is possible to obtain a realistic description of the bottomonium and even the charmonium spectrum with only m_b $[m_c]$ and $\alpha_s(M_Z)$ as input parameters.

Comparison of our potential with lattice calculations

Lattice most reliable in quenched approximation, therefore compare quenched lattice to QCD with $n_f = 0 \rightarrow$ scaleless.

Comparisons pert. QCD ↔ lattice QCD have been made for some time, but with fixed μ and therefore only up to r $0.5r_0$ ($r_0^{-1} = 400\,{\rm MeV}$, i.e. $r_0 \simeq 0.5\,{\rm fm}$)

With the minimum sensitivity scale fixing procedure, the perturbative series remains stable for large values of r:

$\overline{m}$	$\mu = \mu_1$: μ	$E^{(1)}_{\rm tot}$	$E^{(2)}_{\rm tot}$	$E^{(3)}_{\rm tot}$	$E_{\rm tot}$	$\mu = \mu_2$: μ	$E^{(1)}_{\rm tot}$	$E^{(2)}_{\rm tot}$	$E^{(3)}_{\rm tot}$	$E_{\rm tot}$
1.6	0.389	1.275	0.271	−0.280	4.466	0.419	0.921	0.243	0	4.364
1.8	0.413	1.126	0.109	−0.147	4.687	0.449	0.881	0.158	0	4.639
2.0	0.436	1.08	0.038	−0.096	5.022	0.477	0.882	0.111	0	4.993
2.2	0.458	1.073	−0.007	−0.069	5.397	0.502	0.901	0.077	0	5.378
2.4	0.478	1.085	−0.042	−0.051	5.792	0.525	0.929	0.049	0	5.778
2.6	0.497	1.109	−0.072	−0.039	6.197	0.545	0.965	0.022	0	6.187
2.8	0.515	1.140	−0.102	−0.029	6.609	0.563	1.006	−0.005	0	6.601
3.0	0.530	1.179	−0.133	−0.021	7.025	0.576	1.055	−0.035	0	7.019
3.2	0.543	1.224	−0.168	−0.012	7.444	0.453	1.680	−0.682	0	7.398
3.4	0.553	1.277	−0.208	−0.004	7.865	0.507	1.451	−0.393	0	7.859
3.6	0.559	1.342	−0.262	0.006	8.287	0.577	1.288	−0.207	0.005	8.287
3.8	0.554	1.439	−0.352	0.023	8.711	0.615	1.268	−0.173	0.012	8.708

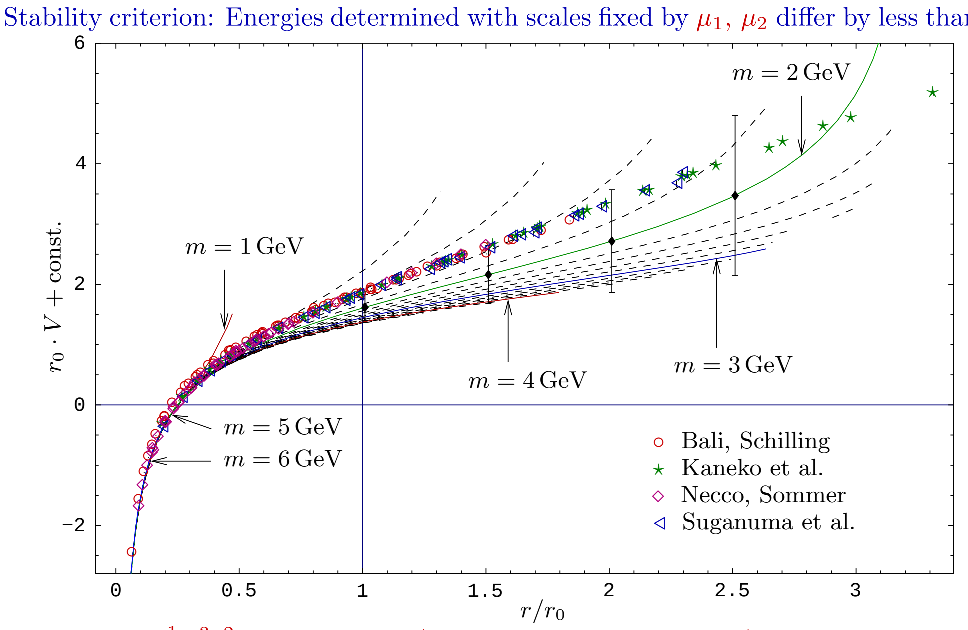
Convergence of $E_{\rm tot}$ for $r = 2r_0 \simeq 5\,{\rm GeV}^{-1}$. All numbers in GeV.

Caution: Comparing perturbative QCD to quenched lattice calculations, therefore no direct connection to real world !

Use Sommer scale: $r^2\,\frac{dV_{\rm QCD}}{dr}\Big|_{r=r_0} = 1.65$, $\Lambda_{\overline{MS}} = 0.602\,r_0^{-1}$ (ALPHA coll.), $r_0^{-1} = 400\,{\rm MeV}$ ($r_0 \simeq 0.5\,{\rm fm}$) for comparison with physical scales

Then: Shift individual sets of lattice data to coincide at r_0, Shift perturbative curves and lattice data to coincide at $r_0/4$. (Uncertainty of perturbative data smaller at $r_0/4$)

Stability criterion: Energies determined with scales fixed by μ_1, μ_2 differ by less than $0.5/r_0$.



Error bars: $\pm\frac{1}{2}\Lambda^3 r^2$, $\Lambda = 300\,{\rm MeV}$ (Next to leading renormalon)

Conclusions:

Minimum sensitivity scale fixing procedure allows stable determinations of QCD potential at rather large distances, in agreement with lattice calculations

This potential can be used to calculate spectra and especially level splittings in heavy quarkonia.

Not completely pure QCD (scale fixing procedure, ...), but **no model parameters !**

Only input is $m_b^{\overline{MS}}$, ($m_c^{\overline{MS}}$,) $\alpha_s(MZ)$.

S. Recksiegel and Y. Sumino, Phys. Rev. D **67**, 014004 (2003)
S. Recksiegel and Y. Sumino, hep-ph/0305178
S. Recksiegel and Y. Sumino, hep-ph/0212389