# QCD SATURATION IN THE SEMI-CLASSICAL APPROACH


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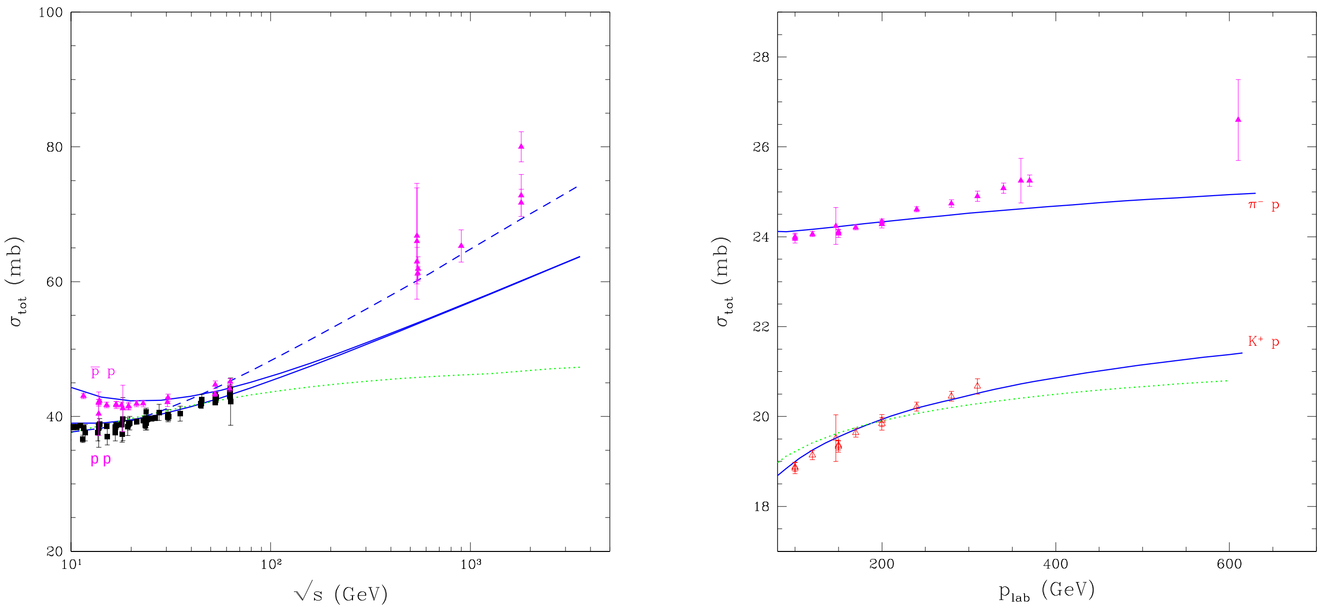
- **Practical needs to understand the large $b_t$ behaviour to solve numerically the non-linear evolution equation;**
- **Theoretical understanding how to incorporate the non-perturbative corrections in the equation at large $b_t$;**
- **Theoretical estimates of the accuracy for the non-linear evolution equation;**
- **Analytical behaviour of the solution in the saturation region;**

## QCD saturation as a part of soft physics

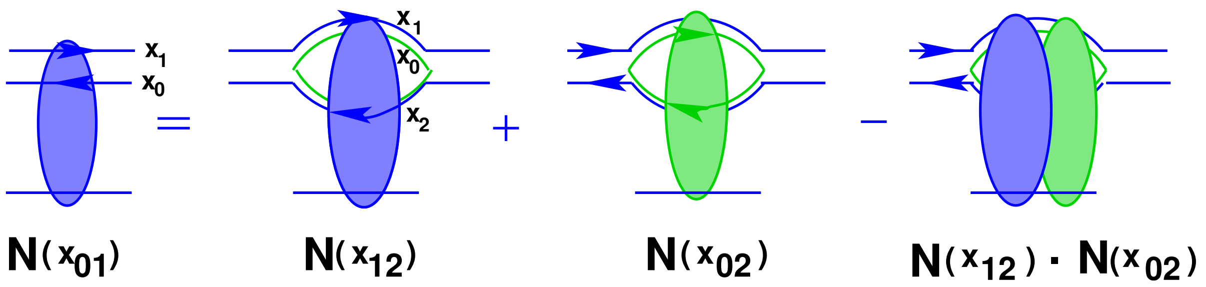
**Saturation $\longrightarrow$ reasonable values and energy behaviour of the soft cross sections**




**Bartels, Lublinsky + TAU**

# Non-linear evolution equation

## Non – Linear colour dipole evolution equation




$$\frac{\partial N(\mathrm{x}_{01}, y; b_t)}{\partial\, y} = -\frac{2\, C_F\, \alpha_S}{\pi}\, \ln\left(\frac{\mathrm{x}_{01}^2}{\rho^2}\right)\; N(\mathrm{x}_{01}, y; b_t) \,+$$

$$\frac{2\, C_F\, \alpha_S}{\pi} \times \int_\rho d^2\mathrm{x}_2 \frac{\mathrm{x}_{01}^2}{\mathrm{x}_{02}^2\, \mathrm{x}_{12}^2} \Big( 2\, N(\mathrm{x}_{02}, y; \mathrm{b_t} - \frac{1}{2}\mathrm{x}_{12})$$
$$- N(x_{02}, y; b_t - \frac{1}{2}x_{12}) N(x_{12}, y; b_t - \frac{1}{2}x_{02}) \Big)$$

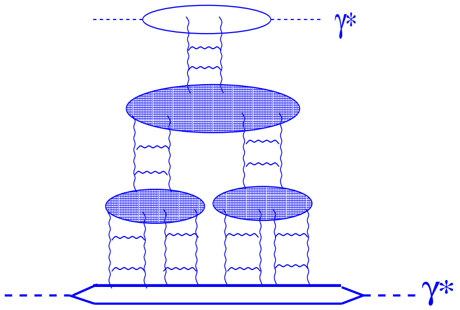
**In momentum representation at large $b_t$**

$$\frac{\partial \tilde{N}(k, y; b_t)}{\partial\, y} \;=\; \hat{\alpha}_S \left( \hat{\chi}(\hat{\gamma}(k))\tilde{N}(k, y; b_t) \;-\tilde{N}^2(k, y; b_t) \right)$$

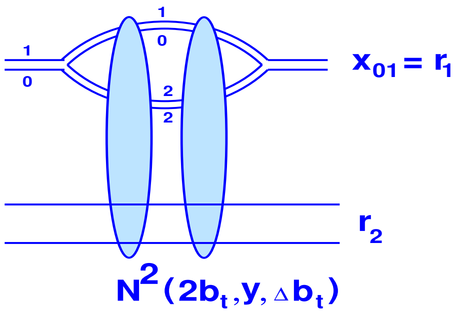
**where**

$$\hat{\gamma}(k) \;=\; 1 \;+\; \frac{\partial}{\partial\, \xi}$$

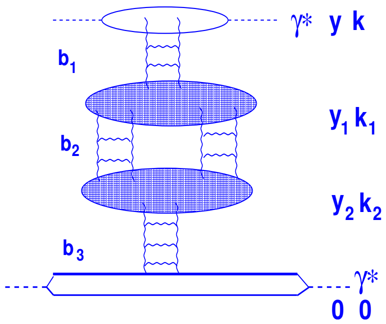
# Accuracy of the non-linear equation

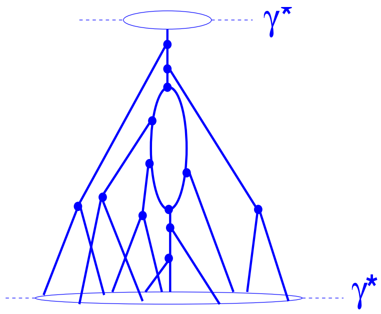



**In the equation the 'fan' diagrams are included:** $r_1 \ll r_2$




**A contribution with** $r_1 > r_2$ **should be treated as corrections**




**Enhanced diagrams are the most important corrections to the non-linear equation**

# BFKL emission at large $b_t$

Emission term in momentum representation

$$\frac{dN(\mathbf{x}_{01}, b_t, y)}{dy} = \boxed{-\frac{2\,C_F\,\alpha_S}{\pi}\,\ln\left(\frac{\mathbf{x}_{01}^2}{\rho^2}\right)\;N(\mathbf{x}, b_t, y)}$$

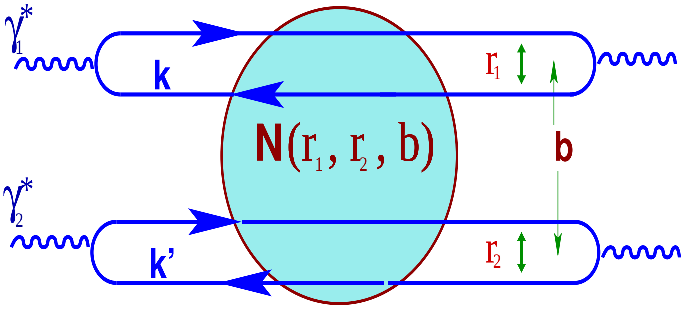
$$+\;\boxed{\frac{2\,C_F\,\alpha_S}{\pi}\int_\rho d^2\mathbf{x}_2\,\boxed{\frac{\mathbf{x}_{01}^2}{\mathbf{x}_{02}^2\,\mathbf{x}_{12}^2}}\cdot N(\mathbf{x}_{02}, b_t, y))}$$

where

$$N(x_{01}, b_t; y) \;=\; \mathrm{Im}\; a_{dipole}(x_{01}, b_t; y)$$

**Gluon Reggeization in momentum representation**

**Probability for dipole –> 2 dipoles decay**




$$N(k, k'; b_t) \;=$$

$$\int \frac{d\gamma}{2\pi i}\,\phi_{in}\,e^{\alpha_S\chi(\gamma)y-(1-\gamma)\xi}$$

Lipatov (1986)

- $\chi(\gamma) = 2\psi(1) - \psi(1-\gamma) - \psi(\gamma)$

- $\xi \;=\; \ln(k^2\,k'^2\,b_t^4)$

- $\phi_{in}(\gamma, k'; b_t)$ **from BA:**

$$\phi_{in}(k', y; b_t) \;=\; \pi\alpha_S^2 \frac{N_c^2 - 1}{3\,N_c^2}\;(m_\pi\,b_t)^4\,K_4(2\,m_\pi\,b_t)\,\frac{1}{\gamma}$$

- **For** $1/2m_\pi \gg\; b_t \;\gg\; 1/k, 1/k'$

$$\phi_{in} \;\;\propto\;\; \frac{1}{b_t^4}$$

- **For** $b_t \;\gg\; 1/2m_\pi$

$$\phi_{in} \;\;\propto\;\; e^{-\,2m_\pi b_t}$$

- **npQCD corrections are included in BA ( Levin & Ryskin (1990), Ferreiro, Iancu, Itakura & McLerran (2002))**

**but not**

**in the equation ( Kovner & Wiedemann (2002))**

# Semi-classical approach ( a general discussion)

- $$N(k, k', y; b_t) = e^S = e^{\omega(\xi, y; b_t)\, y \,-\, (\,1 - \gamma(\xi, y; b_t)\,)\,\xi \,+\, \beta(k', b_t)}$$

- $S_\xi = \gamma \qquad S_y = \omega$

- $\omega(\xi, y; b_t)$ and $\gamma(\xi, y; b_t)$ are smooth function of $y$ and $\xi$

- $$\omega \;-\; \hat{\alpha}_S\,\chi(\gamma) \;+\; \hat{\alpha}_S\,e^S \;=\; 0$$

- **The set of equations on trajectories:**

$$(1.)\ \frac{d\,\xi}{d\,t} \;=\; -\,\hat{\alpha}_S\,\frac{d\,\chi(\gamma)}{d\,\gamma} \quad (2.)\ \frac{d\,y}{d\,t} \;=\; 1 \Longrightarrow \text{trajectories}$$

$$(3.)\qquad \frac{d\,S}{d\,t} \;=\; \hat{\alpha}_S\,(1-\gamma)\,\frac{d\,\chi(\gamma)}{d\,\gamma} + \omega$$

$$(4.)\qquad \frac{d\,\gamma}{d\,t} \;=\; \hat{\alpha}_S\,(\,1 - \gamma\,)\,e^S\;, \qquad (5.)\qquad \frac{d\,\omega}{d\,t} \;=\; -\,\hat{\alpha}_S\,\omega\,e^S$$

$$\frac{d(\gamma - 1)}{(\gamma - 1)} = \frac{d\,\omega}{\omega}$$

$$\omega = B_0\,\omega$$

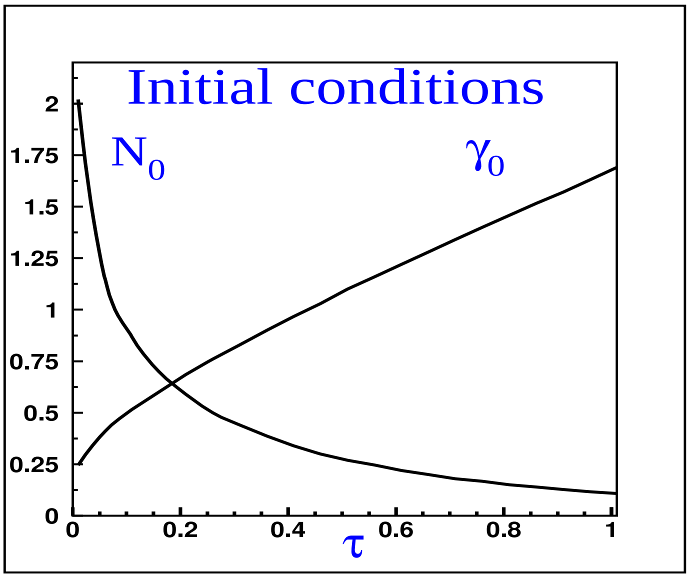
- $$\frac{d\,\gamma}{d\,y} = \alpha_S\,(1 - \gamma)\,(B_0\,(1 - \gamma) + \chi(\gamma))$$

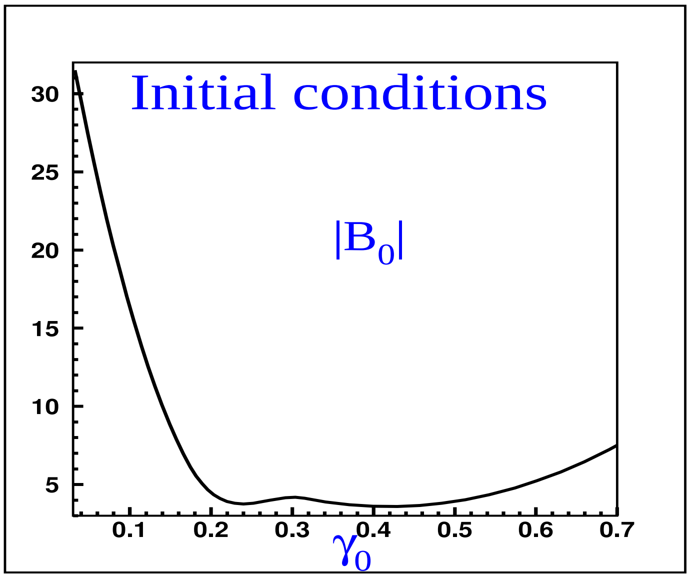
- **Critical trajectory:** (GLR (1982), Bartels & Levin (1992), Mueller & Triantafyllopoulos (2002))

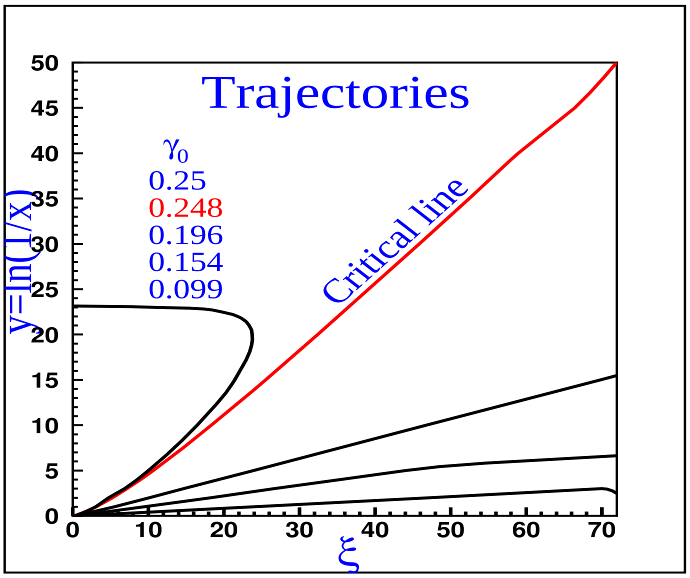
$$\frac{d\,\chi(\gamma_{cr})}{d\,\gamma_{cr}} = -\frac{\chi(\gamma_{cr})}{1 - \gamma_{cr}} \qquad \gamma_{cr} \approx 0.37$$

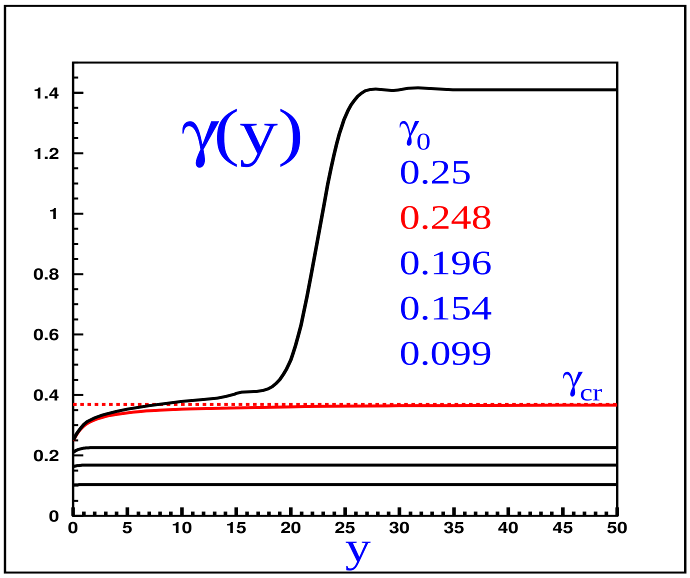
$$\xi_{cr} = -\alpha_S \frac{d\,\chi(\gamma_{cr})}{d\,\gamma_{cr}}(y - y_0) + \xi_0(b_t)$$

$$N = \frac{N_0}{N_0\,\alpha_S\,(y - y_0) + 1}$$


Initial conditions
N0
γ0
τ
0
0.25
0.5
0.75
1
1.25
1.5
1.75
2
0.2
0.4
0.6
0.8


Initial conditions
|B0|
γ0
5
10
15
20
25
30
0.1
0.2
0.3
0.4
0.5
0.6
0.7


Trajectories
γ0
0.25
0.248
0.196
0.154
0.099
Critical line
y=ln(1/x)
ξ
0
5
10
15
20
25
30
35
40
45
50
60
70


γ(y)
γ0
0.25
0.248
0.196
0.154
0.099
γcr
y
0
0.2
0.4
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0.8
1
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45
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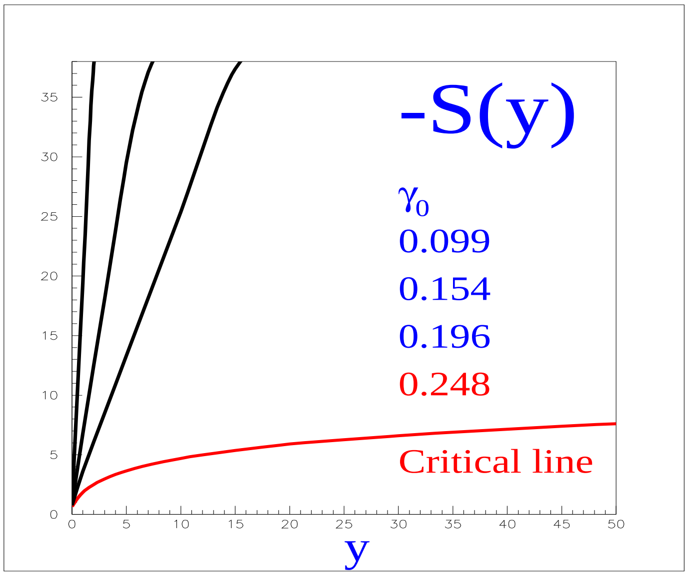

-S(y)
γ0
0.099
0.154
0.196
0.248
Critical line
y
0
5
10
15
20
25
30
35
40
45
50

# $b_t$ dependence

$$\xi \;=\; -\alpha_S \int^{y}_{y_0} \, d\,y' \frac{d\,\chi(\gamma(y'))}{d\gamma} \;+\; \xi_0(b_t) \;.$$

$$Q^2_s(y,k'^2;b_t) \;=\; Q^2_s(y=y_0,k'^2;b_t)\,e^{\alpha_S\,\frac{\chi(\gamma_{cr})}{1-\gamma_{cr}}\,(y-y_0)}$$

$$=\; k'^2\,m^4_\pi\,b^4_t\,K_4(2\,m_\pi\,b_t)\,\tau_0(\gamma=\gamma_{cr})\;e^{\alpha_S\,\frac{\chi(\gamma_{cr})}{1-\gamma_{cr}}\,(y-y_0)}$$

$\Longrightarrow C\;\frac{k'^2}{b^4_t}\,(\frac{1}{x})^\lambda \qquad if\;\; 1/2m_\pi \;>\; b_t \;>\; 1/k'$

$\Longrightarrow C\;\frac{m^4_\pi}{k'^2}\,(\frac{1}{x})^\lambda\,e^{-2\,m_\pi\,b_t} \qquad if\;\; b_t \;>\; 1/2m_\pi$

# Saturation( Colour Glass Condensate) domain

**Geometrical scaling:**

$N = \int \frac{d\omega}{2\pi i} \phi_{in} e^{\omega\, y + \gamma(\omega)\, \xi}$

$\gamma(\omega) = C\,\omega$

for $k^2 \leq Q_s^2(x; b_t)$
$N = N\left(z = \alpha_S \frac{\chi(\gamma_{cr})}{1-\gamma_{cr}}\, y - \xi\right)$

$N = N\left(k^2/Q_s^2(x; b_t)\right)$

**Physics:**

GLR (1982), McLerran & Venugopalan (1994) Stasto, Golec-Biernat& Kwiecinski(2002)

**Theory:**

Bartels & Levin (1994), Ferreiro, Iancu, Itakura & McLerran (2002)

$$N(y, \xi; b_t) = \frac{1}{2} \int_{\xi}^{\xi_{sat}} d\,\xi' \left(1 - e^{-\phi(\xi', y)}\right)$$

$$N(y, \xi; b_t) = N\left(z = \alpha_S \frac{\chi(\gamma_{cr})}{1-\gamma_{cr}}\, y - \xi\right)$$

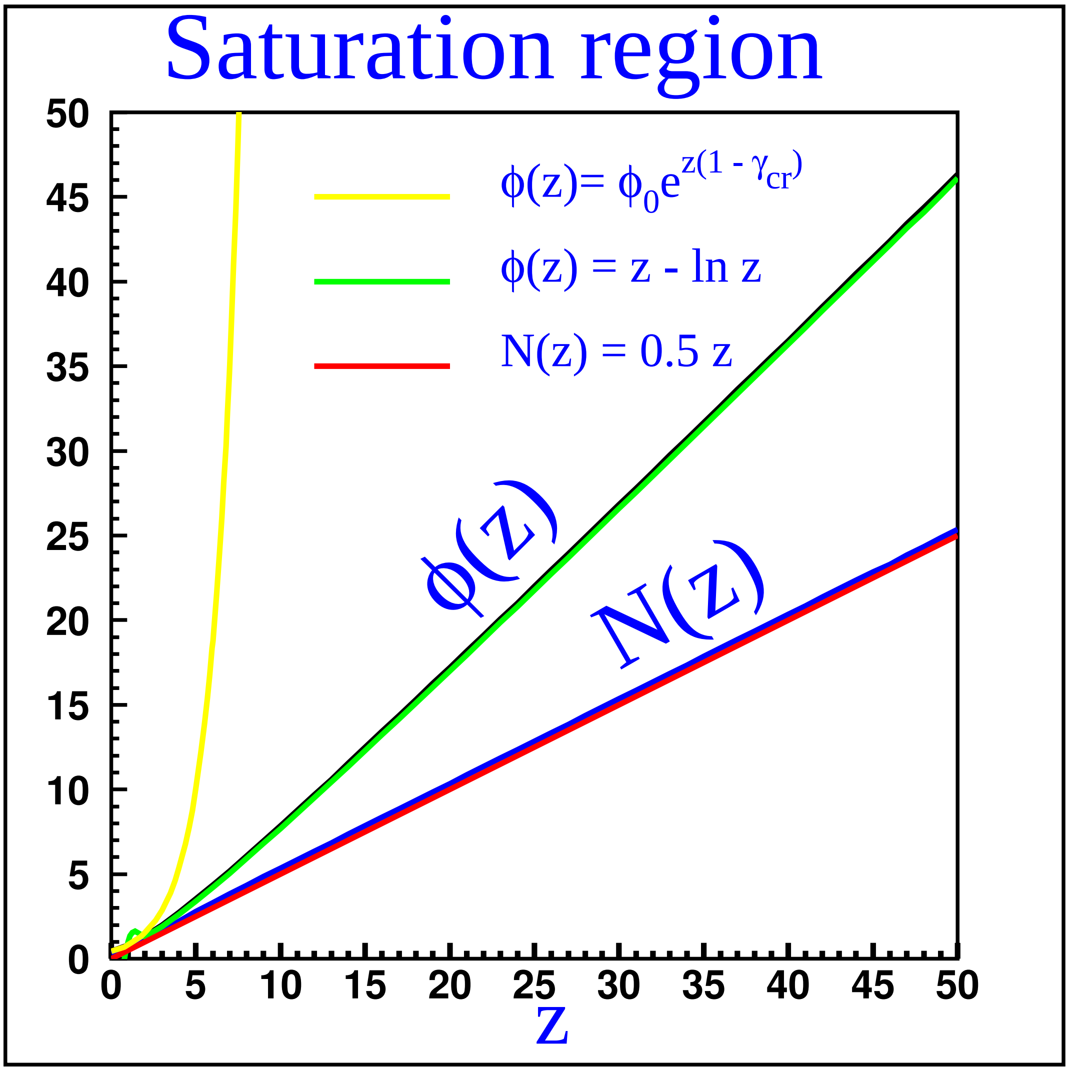
**$\phi$ is a smooth function of $z'$ .**

**Finally:**

$$\frac{d\,\phi(z)}{dz} \,=\, p(z);$$

$$p(z)\,\frac{dp(\phi)}{d\phi} \,+\, \alpha_S\,\frac{dL(p)}{dp}\,p\,\frac{dp(\phi)}{d\phi} \,=\, \alpha_S\,\left(1 \,-\, e^{-\,\phi}\right)\,;$$

$$L(f) \,=\, \frac{(1-f)\,\chi(f) \,-\, 1}{f};$$



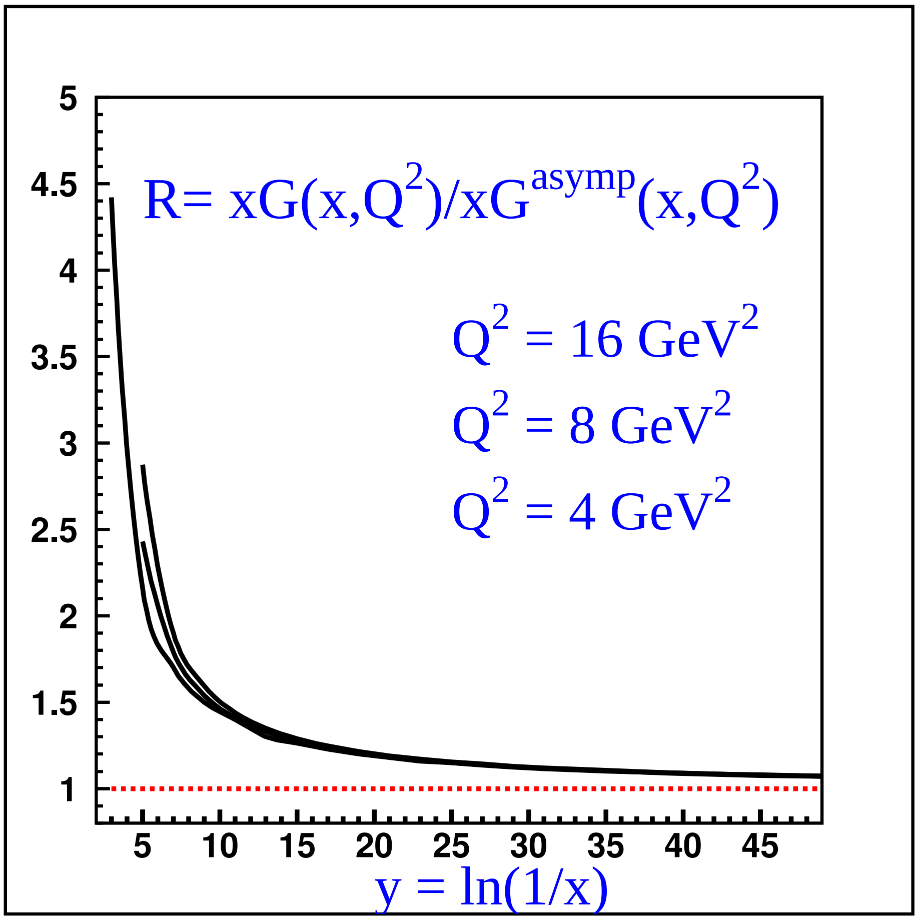

$$\sigma_{tot}(dipole) = 2 \int d^2 b_t \, N(z = \ln(Q_s^2(y; b_t) \; - \; \xi)$$

$$= 2 \int d^2 b_t \, N(z = \ln(Q_s^2(y; b_t = 0)) \; - 2 m_\pi \, b_t \; - \; \xi)$$

$$\Longrightarrow |_{x \to 0} \quad \frac{2\pi}{3(2m_\pi)^2} \ln^3(Q_s^2(x)/k^2) \;\; \propto \;\; \boxed{\frac{2\pi}{3(2m_\pi)^2} \ln^3(1/x)}$$

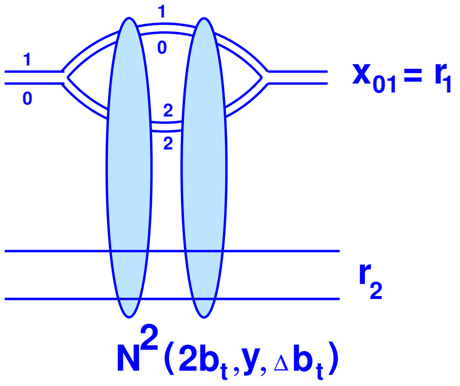
$$xG(x,Q^2) \;=\; \frac{4N_c\pi}{\alpha_S}\int^{Q^2} d\,k^2\sigma \;\propto\; \frac{Q^2}{(2m_\pi)^2}\,\ln^3 Q_s^2(x)$$

$$\Longrightarrow\quad \frac{Q^2}{(2m_\pi)^2}\,\ln^3(1/x)$$



# Accuracy of the approach

- Contribution due to interaction of two large dipoles $x_{12} \approx x_{02} \approx 2\,b_t \gg r_2$

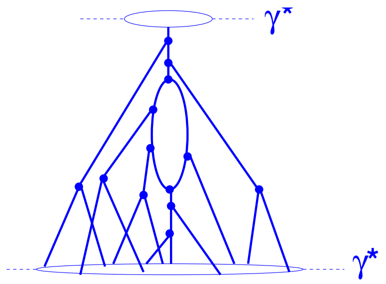



- $\Delta N =$ $\alpha_S(x_{01}) \int d^2\Delta b_t \; \frac{x^2_{01}}{(b_t + \Delta b_t)^4}$ $N^2(2(b_t + \Delta b_t), y, \Delta b_t)$

- For $b_t \gg b_0(y)$ $\Delta N \approx \alpha_S(x_{01}) \frac{x^2_{01}\, b^2_0(y)}{b^4_t}$;

- For $r_2 \ll b_t \ll b_0(y)$ $\Delta N \approx \alpha_S(x_{01}) \, \frac{x^2_{01}}{b^2_0(y)}$ .

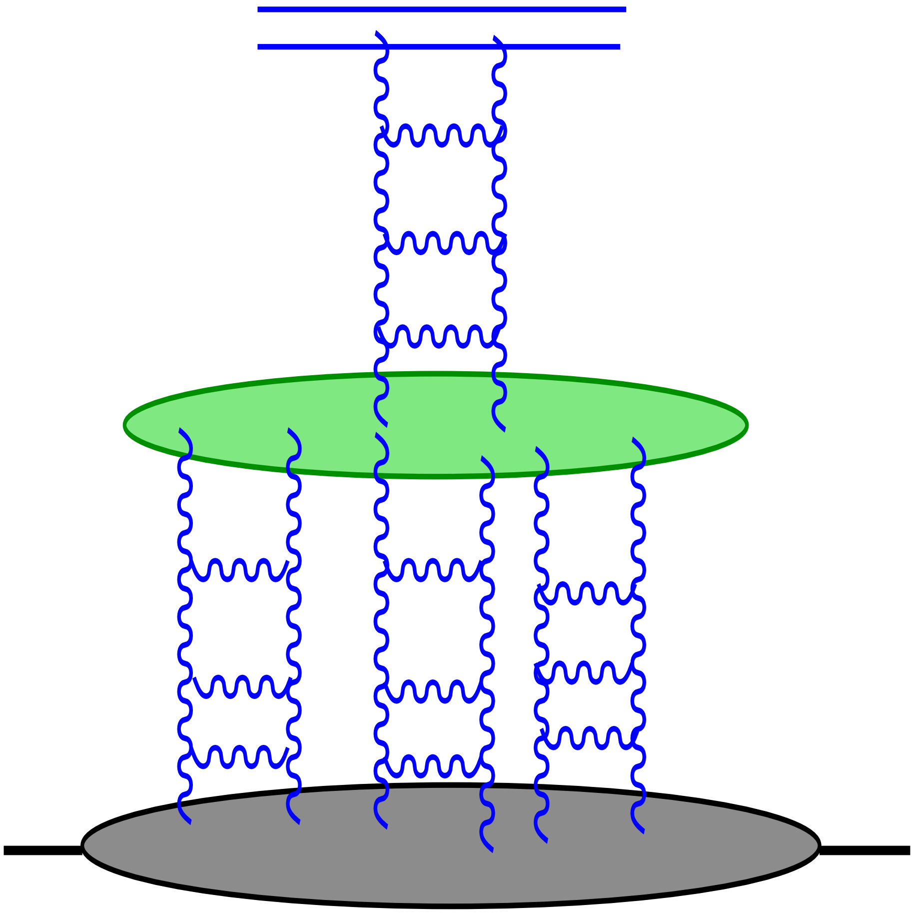
- Enhanced diagrams




- $\int d^2 b_3 \Delta N^{enhance} = \frac{\hat{\alpha}^2_S}{N^2_c} \int d^2 b_3 \int_0^y dy_1$ $\int d^2 k_1\, d^2 b_1 \int_0^{y_1} dy_2 \int d^2 k_2 d^2 b_2$ $\tilde{N}(k, k_1, y - y_1, b_1)$ $\tilde{N}^2(k_1, k_2, y_1 - y_2, b_2)\tilde{N}(k_1, k', y_2, b_3)$

$$\boxed{\propto \quad \frac{\hat{\alpha}^2_S}{N^2_c}\, y^2}$$

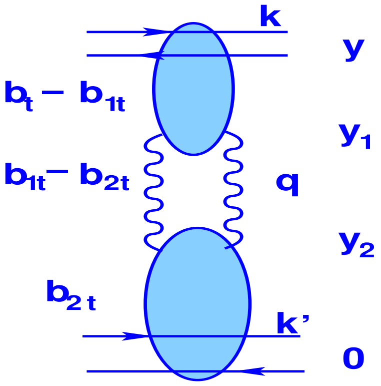
- **A new non-linear equation is needed (?)**



**= ? ? ?**

♠ **First attempt to answer:**
**Levin & Lublinsky hep-ph/0308279**

- **Why npQCD only in initial condition?**




$$\tilde{N}(k,k',y;b_t) \;=\; \frac{1}{b_t^2}$$
$$\int d^2q \int_0^y dy_1 \int d^2(b_t - b_{1t})$$
$$\tilde{N}(k,q,y-y_1;b_t - b_{1t})$$
$$\int_0^{y_1} dy_2 \int d^2(b_t - b_{1t})$$
$$\tilde{N}(k,q,y_2;b_{2t})$$

$$\boxed{\tilde{N}(k,k',y;b_t) \longrightarrow \frac{1}{b_t^2\,(2m_\pi)^2} \int_0^y dy_1 F(y-y_1) \int_0^{y_1} dy_2\, y_2^2.}$$

# Running $\alpha_S$

1. $\frac{d\,\xi}{d\,t} = F_\gamma \; = \; - \; \hat{\alpha}_S(\xi)\, \frac{d\,\chi(\gamma)}{d\;\gamma}$

2. $\frac{d\,y}{d\,t} = F_\omega = 1 \;;$

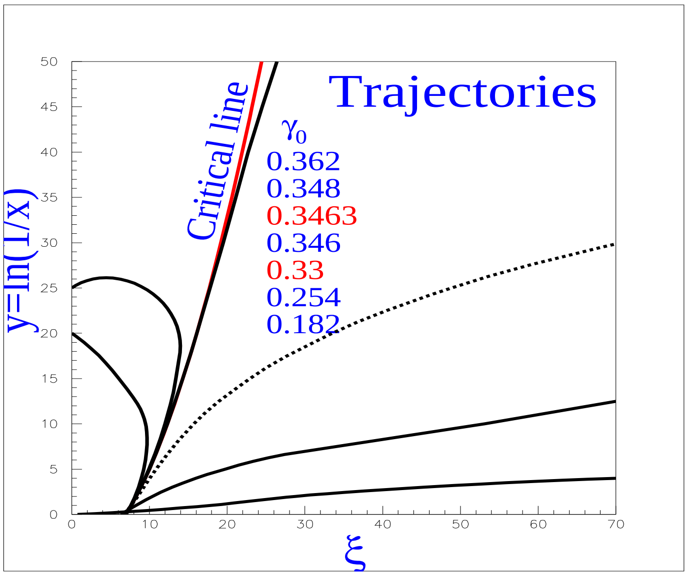
3. $\frac{d\,S}{d\,t} = \gamma\,F_\gamma + \omega\,F_\omega = \hat{\alpha}_S(\xi)\,(1-\gamma)\,\frac{d\,\chi(\gamma)}{d\,\gamma} + \omega\;;$

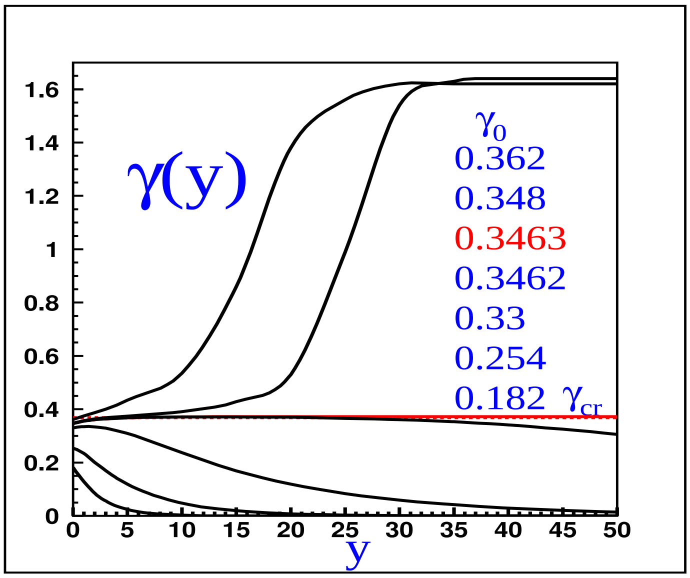
4. $\frac{d\,\gamma}{d\,t} = \; - \; (F_\xi \; + \; \gamma\,F_S)$

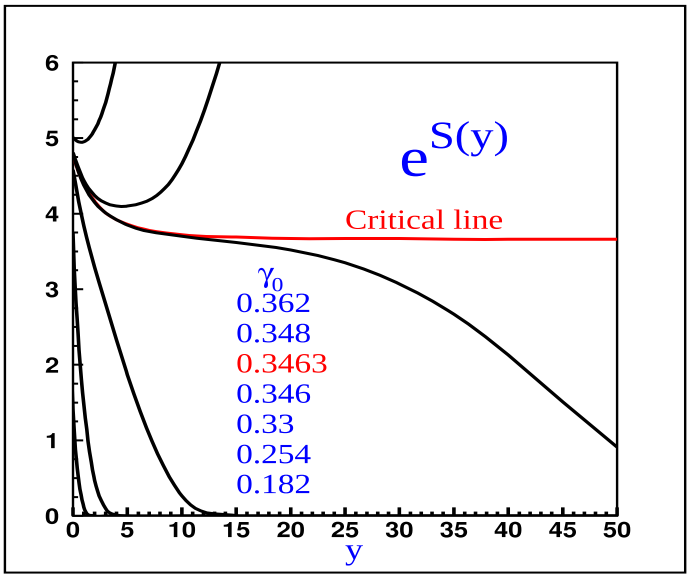
$= \hat{\alpha}^2_S(\xi)\,(\,1-\gamma\,)\,e^S \; - \; \hat{\alpha}^2_S(\xi)\,\frac{b}{4\,\pi}\,\chi(\gamma) \; + \; \hat{\alpha}^3_S(\xi)\,\frac{b}{2\pi}\,e^S\;;$

5. $\frac{d\,\omega}{d\,t} = \; - \; (F_y \; + \; \omega\,F_S) \;\; = \; - \; \hat{\alpha}^2_S(\xi)\,\omega\,e^S\;;$

# Running $\alpha_S$




## Saturation momentum:

$$(\xi - \bar{\xi}(k', b_t))^2 - (\,\xi_0(b_t) - \bar{\xi}(k', b_t)\,)^2 = \frac{4\,\pi}{b}\,\frac{d\chi(\gamma_{cr})}{d\gamma_{cr}}\,(y - y_0)\ ,$$

# Running $\alpha_S$

$$Q_s^2(y,b_t) \;=\; \Lambda^2\, e^{\sqrt{(\,\xi_0(b_t)\;-\;\bar{\xi}(k',b_t)\,)^2\;+\;\frac{4\,\pi}{b}\chi'_{\gamma_{cr}}(\gamma_{cr})\,(y-y_0\,)}}$$

$Q_s$ **does not depend on** $b_t$ **for**

$$b_t\;\leq\;(1/2m_\pi)\,\sqrt{\frac{4\,\pi}{b}\chi'_{\gamma_{cr}}(\gamma_{cr})\,(y\;-\;y_0\,)}$$

**BUT for**

$$b_t\;\geq\;(1/2m_\pi)\,\sqrt{\frac{4\,\pi}{b}\chi'_{\gamma_{cr}}(\gamma_{cr})\,(y\;-\;y_0\,)}$$

$$Q_s^2\;\longrightarrow\;e^{-2\,m_\pi\,b_t}$$

# Conclusions.

**1.** Recommendation for searching of numeric solution:

$$r(\text{dipole}) \leq R(\text{target})$$

**2.** Solution in the saturation ( Colour Glass Condensate ) domain:

$$N\left(r_t^2 Q_s^2(x; b_t)\right)$$

$\alpha_S = \text{Const}:$

$$\sigma_{dipole}(r_t, x) \leq \frac{2\,\pi}{(2\,m_\pi)^2}\ \ln^2\left(Q_s^2(x)\,r_t^2\right)$$

$$\sigma_{dipole}(r_t, x) \; \leq \; \frac{2\,\pi}{(2\,m_\pi)^2}\,\left(\alpha_S \frac{\chi(\gamma_{cr})}{1-\gamma_{cr}}\,\ln(1/x)\right)^2$$

# BUT NOT

$$\sigma_{dipole}(r_t, x) \; \leq \frac{2\,\pi}{(2\,m_\pi)^2}\,\left(\alpha_S \chi(\frac{1}{2})\,\ln(1/x)\right)^2$$

**E.L. & Ryskin (1990); Ferreiro, Iancu, Itamura & McLerran (2002)**

**Running $\alpha_S$ :**

$$\sigma_{dipole}(r_t, x) \; \leq \; \frac{2\,\pi}{(2\,m_\pi)^2}\,\ln^2\left(Q^2_s(x)\,r^2_t\right)$$

$$\sigma_{dipole}(r_t, x) \; \leq \; \frac{2\,\pi}{(2\,m_\pi)^2}\,\left(\frac{4\,\pi\,\chi(\gamma_{cr})}{b\,(1-\gamma_{cr})}\,\ln(1/x)\right)$$

# BUT NOT

$$\sigma_{dipole}(r_t, x) \;\leq\; \frac{2\,\pi}{(2\,m_\pi)^2}\,\left(\alpha_S\,\chi(\frac{1}{2})\,\ln(1/x)\right)^2$$

**E.L. & Ryskin (1990);**
**Ferreiro, Iancu, Itamura & McLerran (2002)**

**3.** We found that all corrections are small and the real restrictions for the non-linear equation stems from the contribution of the enhanced diagrams, namely,

$$\alpha_S\,ln^2(1/x) \quad \leq \quad N_c^2$$

**4.** We hope that the analytic solution in the saturation region will be useful in understanding of the behaviour of the high energy scattering amplitude .