Hamiltonian analyze of the particle motion in an accelerator with the longitudinal magnetic field.

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Abstract

The particle motion at a presence of a large magnetic field directed along the particle trajectory demands the special description. This article deals with the decomposition of the Hamiltonian on the two parts: fast and slow motion. The first part describes the fast rotation around the magnetic line of longitudinal field. The second part describes the slow drift of rotation center from one magnetic line to another. The supposed method enables to write the simple Hamiltonian to each motion type and to formulate the matrix formalism for any element of an accelerator device (quadruple, skew- quadruple, drift gap, bend with a field index). The Hamiltonian decomposition has physical clearness when the longitudinal field is larger than another fields but it is correct for the arbitrary parameters. At the small longitudinal field the coupling term in Hamiltonian between two modes is essential. The dispersion property of fast and slow modes is derived easy from Hamiltonian also. This method expands easily for nonlinear motion of such modes. This results may be used at analyzed the electron motion in the cooling device, the muon motion in the muon ionization cooler [1] or another system with strong solenoidal coupling [2].

Hamiltonian method

A non-relativistic Hamiltonian for a single particle in the electromagnetic field can be written as [3]

$$H=\frac{1}{2m}\left(P_x-\frac{e}{c}A_x\right)^2+\frac{1}{2m}\left(P_y-\frac{e}{c}A_y\right)^2+\frac{1}{2m}\left(P_z-\frac{e}{c}A_z\right)^2 \qquad (1)$$

where P_x, P_y, P_z are canonical momenta conjugated to the coordinates X,Y,Z. To describe the particle motion along the reference orbit Frenet coordinate system is used. A particle position is written as

$$\vec r=\vec r_0(s)+x\vec n(s)+y\vec b(s)\,, \qquad (2)$$

where $\vec r_0(s)$ is the position vector of the particle, $\vec\tau(s)$, $\vec n(s)$ and $\vec b(s)$ are tangent, normal and binormal vectors to the reference orbit. The transverse displacement of the particle from the reference orbit is described by coordinates x and y.

Deriving the canonical transformation with generating function

$$F=-\vec P\cdot\left[\vec r_0+x\vec n(s)+y\vec b(s)\right], \qquad (3)$$

where $P=(P_x,P_y,P_z)$ is old canonical momenta and (x,y,s) is new coordinates, one can write new Hamiltonian as

$$H=\frac{1}{2m}\left[\left(q_x-\frac{e}{c}A_1\right)^2+\left(q_y-\frac{e}{c}A_2\right)^2+\frac{\left(q_s-\frac{e}{c}A_3\right)^2}{(1+hx)^2}\right]. \qquad (4)$$

Here (q_x, q_y, q_z) is new canonical momenta, $h=1/\rho$ is the curvature of the reference orbit, ρ is the radius of curvature, ($A_1=\vec A\cdot\vec n$, $A_2=\vec A\cdot\vec b$, $A_3=\vec A\cdot\vec\tau\,(1+hx)$) is the vector potential of the magnetic field.

The magnetic field expansion around the reference orbit is taken as [4]

$$\begin{aligned}
A_x(x,y,z)&=-\frac{1}{2}b_sy+\frac{1}{3}hb_sxy-\frac{1}{3}b_0'y^2-\\
&-\frac{1}{8}\left(2h^2b_s+a_1'\right)x^2y-\frac{1}{4}\left(b_1'-hb_0'\right)xy^2+\frac{1}{8}\left(a_1'+b_s''\right)y^3+\dots,\\
A_y(x,y,z)&=\frac{1}{2}b_sx-\frac{1}{3}hb_sx^2+\frac{1}{3}b_0'xy+\\
&+\frac{1}{8}\left(2h^2b_s+a_1'\right)x^3+\frac{1}{4}\left(b_1'-hb_0'\right)x^2y-\frac{1}{8}\left(a_1'+b_s''\right)xy^2+\dots\\
A_s(x,y,z)&=-b_0x-\frac{1}{2}(b_1-hb_0)x^2+\frac{1}{2}\left(2a_1+b_s'\right)xy+\frac{1}{2}b_1y^2-\\
&-\frac{1}{6}\left(2b_2-hb_1+3h^2b_0\right)x^3+\frac{1}{6}\left(6a_2-5hb_s'-2h'b_s\right)x^2y+\\
&+\frac{1}{6}\left(6b_2+b_0''\right)xy^2-\frac{1}{6}\left(2a_2+h\left(a_1-2b_s'\right)-h'b_s\right)y^3.
\end{aligned} \qquad (5)$$

The independent components a_n and b_n are the multipole coefficients of the transverse magnetic field on the reference orbit

$$\left.\frac{\partial^nB_y}{\partial x^n}\right|_{x=y=0}=n!\cdot b_n,\quad \left.\frac{\partial^{n-1}B_x}{\partial x^{n-1}}\right|_{x=y=0}=(n-1)!\cdot a_{n-1}\,, \qquad (6)$$

b_s is the longitudinal magnetic field. Keeping only the linear and quadratic terms the magnetic field can be read as

$$\begin{aligned}
A_x&=-\frac{1}{2}B_s(s)y,\\
A_s&=-b_0(s)x-\frac{1}{2}\left(b_1(s)-h(s)b_0(s)\right)x^2+\left(a_1(s)+\frac{1}{2}B_s'(s)\right)xy+\frac{1}{2}b_1(s)y^2.
\end{aligned} \qquad (7)$$

Here one should take into account that no uniform longitudinal magnetic field gives the contribution in term $a_1(s)$ according to equation (6). For example, the straight gap with changing longitudinal magnetic field results in

$$a_1=\frac{\partial B_x}{\partial x}=-\frac{1}{2}B_s'\,. \qquad (8)$$

In linear approximation for the particle motion it is possible to use this equation in the bending section. It is necessary to use the accurate paraxial expansion of the magnetic field for the approximation of the higher order.

The resulting Hamiltonian of the particle motion is

$$\begin{aligned}
H&=\frac{p_s^2}{2}+\frac{p_x^2}{2}+\frac{p_y^2}{2}+\frac{x^2}{2}\left(K_x^2+K+\frac{R^2}{4}\right)-N\,x\cdot y-\\
&-K_xp_sx-\frac{R}{2}\left(p_yx-p_xy\right)+\frac{y^2}{2}\left(-K+\frac{R^2}{4}\right),
\end{aligned} \qquad (9)$$

Here $\left(B_x,B_y,B_s\right)$ is the horizontal, vertical and longitudinal components of the magnetic fields, $K_x=\frac{e\,B_y}{q_{s0}\,c}$, $R=\frac{e\,B_s}{q_{s0}\,c}$, $K=K_x\frac{1}{B_y}\frac{\partial B_y}{\partial x}$ is normal component of the magnetic field gradient, $N=K_x\frac{1}{B_y}\frac{\partial B_x}{\partial x}$ is skew component of the magnetic field gradient, $N_s=K_x\frac{1}{B_y}\frac{\partial B_s}{\partial s}$ is longitudinal gradient of the longitudinal magnetic field. All parameters of motion K, K_x, R, N, N_s are function of longitudinal position and time as $f(s+\xi)$. The longitudinal and transverse momenta are normalized on the total particle momentum.

At condition of the strong longitudinal magnetic field the motion can be decompose on the fast Larmour rotation around the magnetic force line and slow drift of Larmour center. The center of the Larmour rotation moves along the magnetic field force line and drifts slowly in the plane (x,y). This drift motion is induced by the small transverse component of the magnetic and electrical forces, non-homogeneity of the longitudinal magnetic field or the centrifugal force. This physical picture is convenient if the longitudinal magnetic force is strong.

Let us to do the change of variables

$$\left[\begin{aligned}P_1&=p_x-\frac{1}{2}Ry\\ Q_1&=\frac{p_y}{R}+\frac{1}{2}x\end{aligned}\right.,\qquad \left[\begin{aligned}P_2&=p_y-\frac{1}{2}Rx\\ Q_2&=\frac{p_x}{R}+\frac{1}{2}y\end{aligned}\right.. \qquad (10)$$

Taking into account that p_x and p_y are the canonical momenta $\vec p=\vec\rho-\frac{e}{c}\vec A$ one can obtain the correlation between the new variable and usual momenta (ρ_x, ρ_y) and coordinates (x, y)..

$$\left[\begin{aligned}P_1&=\frac{\rho_x}{p_{s0}}-\frac{e\,B_s}{p_{s0}\,c}y\\ Q_1&=\frac{p_{s0}\,c}{e\,B_s}\frac{\rho_y}{p_{s0}}+\frac{x}{2}\end{aligned}\right.,\qquad \left[\begin{aligned}P_2&=\frac{\rho_y}{p_{s0}}\\ Q_2&=\frac{p_{s0}\,c}{e\,B_s}\frac{\rho_x}{p_{s0}}\end{aligned}\right.. \qquad (11)$$

The mode (P_1,Q_1) describes the coordinates (X,Y) of the center of the Larmour circle. The mode (P_2,Q_2) relates to the rotation amplitudes of the particle around the magnetic force line.

The generating function for this variables changing is

$$\Psi(x,y,Q_1,Q_2)=R\left(-\frac{1}{2}x\,y-Q_1Q_2+xQ_2+yQ_1\right). \qquad (12)$$

The new Hamiltonian is

$$\begin{aligned}
H&=-\frac{K}{R^2}\frac{P_1^2}{2}+\left(K_x^2+K\right)\frac{Q_1^2}{2}+\frac{1}{R}\left(N-\frac{N_s}{2}\right)P_1Q_1+\\
&+\left(1+\frac{K_x^2}{R^2}+\frac{K}{R^2}\right)\frac{P_2^2}{2}+\left(R^2-K\right)\frac{Q_2^2}{2}+\frac{1}{R}\left(N-\frac{N_s}{2}\right)P_2Q_2-\\
&-\frac{1}{R}\left(K_x^2+K\right)P_2Q_2+\frac{K}{R}P_1Q_2-N\left(Q_1Q_2+\frac{1}{R^2}P_1P_2\right)+\frac{N_2}{2}\left(Q_1Q_2-\frac{1}{R^2}P_1P_2\right)+\\
&+\frac{p_s^2}{2}+\frac{K_x}{R}P_2p_s-K_xQ_1p_s
\end{aligned} \qquad (13)$$

The first string describes the motion of the slow mode (P_1,Q_1) with large "pseudo"-mass $M=R^2/K$. The second string describes the fast oscillation of the mode (P_2,Q_2) with frequency R. The third string is coupling between modes (P_1,Q_1) and (P_2,Q_2). In the case $R>>K$, K_x, N, N_s it is small and can be consider with perturbation method. The last string is the longitudinal motion and dispersion terms.

Drift section

This case the Hamiltonian deals with the particle motion in the simple straight solenoid ($K=0$, $K_x=0$, $N=0$, $N_s=0$)

$$H=\frac{1}{2}p_s^2+\frac{1}{2}P_2^2+\frac{R^2}{2}Q_2^2 \qquad (14)$$

The Hamiltonian doesn't depends from variable P_1 and Q_1 . Thus, P_1 and Q_1 are the motion integral and the center of the Larmour circle is immovable. The fast rotation is described by the standard Hamiltonian of an oscillator.

The matrix for the slow motion is

$$\begin{pmatrix}\tilde Q_1\\ \tilde P_1\end{pmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}\cdot\begin{pmatrix}Q_1\\ P_1\end{pmatrix}. \qquad (15)$$

Here $\left(P_1,Q_1\right)$ are variables of slow mode before the element and $\left(\tilde P_1,\tilde Q_1\right)$ are variables after element of drift section.

Variation of the longitudinal magnetic field.

The Hamiltonian for the straight section with longitudinal magnetic field changing can be read as

$$H=\frac{1}{2}p_s^2+\frac{1}{2}P_2^2+\frac{R^2}{2}Q_2^2-\frac{1}{2}\frac{N_s}{R}P_2Q_2-\frac{1}{2}\frac{N_s}{R}P_1Q_1+\frac{N_s}{2}\left(Q_1Q_2-\frac{1}{R^2}P_1P_2\right) \qquad (16)$$

Ignoring the coupling term one can write

$$H=\frac{1}{2}p_s^2+\frac{1}{2}P_2^2+\frac{R^2}{2}Q_2^2-\frac{1}{2}\frac{N_s}{R}P_1Q_1. \qquad (17)$$

The equations for the motion of the slow mode are

$$\begin{aligned}
P_1'&=-\frac{\partial H}{\partial Q_1}=\frac{N_s}{2R}P_1=\frac{1}{2B_s}\frac{\partial B_s}{\partial s}P_1\\
Q_1'&=\frac{\partial H}{\partial P_1}=-\frac{N_s}{2R}Q_1=-\frac{1}{2B_s}\frac{\partial B_s}{\partial s}Q_1
\end{aligned}. \qquad (18)$$

From this equation one can see that

$$Y\,B_s^{1/2}=const \text{ and } X\,B_s^{1/2}=const\,.$$

So the magnetic flux through the beam cross section is constant.

The motion of the fast mode is described by the Hamiltonian

$$H=\frac{1}{2}P_2^2+\frac{R^2}{2}Q_2^2. \qquad (19)$$

The slow changing of the frequency of the driving force conserves the adiabatic invariant, so

$$\frac{H}{R}=const \;\Rightarrow\; \frac{\rho_x^2}{B_s}=const \quad\text{and}\quad \frac{\rho_y^2}{B_s}=const\,. \qquad (20)$$

The corresponding matrix for this optics element is

$$\begin{pmatrix}\tilde Q_1\\ \tilde P_1\end{pmatrix}=\begin{bmatrix}1/\sqrt{B_s}&0\\0&\sqrt{B_s}\end{bmatrix}\cdot\begin{pmatrix}Q_1\\ P_1\end{pmatrix}.$$

Quadruple lens with longitudinal magnetic field

The total Hamiltonian with coupling term is

$$H=-\frac{K}{R^2}\frac{P_1^2}{2}+K\frac{Q_1^2}{2}+\left(1+\frac{K}{R^2}\right)\frac{P_2^2}{2}+\left(R^2-K\right)\frac{Q_2^2}{2}-\frac{K}{R}\left(P_2Q_1-Q_2P_1\right)+\frac{p_s^2}{2} \qquad (21)$$

and it can be read after simplification as

$$H=-\frac{K}{R^2}\frac{P_1^2}{2}+K\frac{Q_1^2}{2}+\frac{P_2^2}{2}+R^2\frac{Q_2^2}{2}+\frac{p_s^2}{2}. \qquad (22)$$

The fast rotation doesn't change significantly. The motion of the Larmour center can be described as

$$\begin{aligned}P_1'&=-KQ_1\\ Q_1'&=-\frac{K}{R^2}P_1\end{aligned}\quad ==> \quad \begin{aligned}\frac{\partial}{\partial\xi}\left(P_1+RQ_1\right)&=-\frac{K}{R}\left(P_1+RQ_1\right)\\ \frac{\partial}{\partial\xi}\left(P_1-RQ_1\right)&=\frac{K}{R}\left(P_1-RQ_1\right)\end{aligned}. \qquad (23)$$

The beam is reshaped to an ellipse with axes tilted on angle 45° to (x,y) coordinate system. Along one axis the beam is stretched, along other axis is compressed. The particle motion in skew quadruple lens is analogously.

The simplify Hamiltonian of motion is

Hamiltonian analyze of the particle motion in an accelerator with the longitudinal magnetic field.

$$H = \frac{N}{R} P_1 Q_1 + \frac{P_2^2}{2} + R^2 \frac{Q_2^2}{2} + \frac{p_s^2}{2}. \qquad (24)$$

and the equation of the particle motion is

$$\frac{\partial}{\partial \xi} P_1 = -\frac{N}{R} P_1, \quad \frac{\partial}{\partial \xi} Q_1 = \frac{N}{R} Q_1 . \qquad (25)$$

The matrix of elements for slow mode are

$$\begin{pmatrix} \tilde{Q}_1 \\ \tilde{P}_1 \end{pmatrix} = \begin{bmatrix} \cosh\left(\frac{K}{R}s\right) & -\frac{1}{R}\sinh\left(\frac{K}{R}s\right) \\ -R\sinh\left(\frac{K}{R}s\right) & \cosh\left(\frac{K}{R}s\right) \end{bmatrix} \cdot \begin{pmatrix} Q_1 \\ P_1 \end{pmatrix} \text{ (quadrupole)}, \qquad (26)$$

and

$$\begin{pmatrix} \tilde{Q}_1 \\ \tilde{P}_1 \end{pmatrix} = \begin{bmatrix} \exp\left(-\frac{K}{R}s\right) & 0 \\ 0 & \exp\left(\frac{K}{R}s\right) \end{bmatrix} \cdot \begin{pmatrix} Q_1 \\ P_1 \end{pmatrix} \text{ (skew-quadrupole)} . \qquad (27)$$

Bending magnet

The particle moves in the bending magnet with some field index. Along the particle trajectory the longitudinal magnetic field is applied. The Hamiltonian of such motion is

$$H = -\frac{K}{R^2}\frac{P_1^2}{2} + \left(K_x^2 + K\right)\frac{Q_1^2}{2} + \frac{P_2^2}{2} + R^2\frac{Q_2^2}{2} + \frac{p_s^2}{2} + \frac{K_x}{R} P_2 p_s - K_x Q_1 p_s . \qquad (28)$$

At $K < 0$ the dynamic of the mode (P_1, Q_1) is similar to some oscillator. The centers of the Larmour circles move along an ellipse curve. At the field index *n=0.5* this curve is circle. The length corresponding one turn along this curve is

$$L = 2\pi \frac{R^2}{|K|\left(K_x^2 - |K|\right)} \qquad (29)$$

In the case $K = 0$ the field index n=0 and particle motion is

$$P_1' = K_x^2 Q_1, \quad Q_1' = 0 \;\Rightarrow\; X = const, \quad Y' = \frac{K_x^2}{R} X . \qquad (30)$$

Thus, the horizontal position of the particle is constant. At nonzero horizontal shift the centrifugal force doesn't balanced by the bending magnetic field B_y. As result the particle has unlimited shift in the vertical direction.

The matrix of element for slow mode is

$$\begin{pmatrix} \tilde{Q}_1 \\ \tilde{P}_1 \end{pmatrix} = \begin{bmatrix} \cos(\lambda s) & \frac{|K|}{\lambda R^2}\sin(\lambda s) \\ -\frac{\lambda R^2}{|K|}\sin(\lambda s) & \cos(\lambda s) \end{bmatrix} \cdot \begin{pmatrix} Q_1 \\ P_1 \end{pmatrix}, \quad \lambda = \frac{\sqrt{|K|\left(K_x^2 - |K|\right)}}{R} \qquad (31)$$

Taking into account the dispersion terms the Hamiltonian can be read as

$$H = -\frac{K}{R^2}\frac{P_1^2}{2} + \frac{1}{2}(K_x^2 + K)\left(Q_1 - \frac{K_x}{K_x^2 + K} p_s\right)^2 + \frac{1}{2}\left(p_2 + \frac{K_x}{R} p_s\right)^2 + R^2\frac{Q_2^2}{2} + \frac{P_s^2}{2}\left(\frac{K}{K_x^2 + K} - \frac{K_x^2}{R_2}\right) \qquad (32)$$

or ignoring the term $K_x^2/R^2 << 1$

$$H = -\frac{K}{R^2}\frac{P_1^2}{2} + \frac{1}{2}(K_x^2 + K)\left(Q_1 - \frac{K_x}{K_x^2 + K} p_s\right)^2 + \frac{1}{2}\left(p_2 + \frac{K_x}{R} p_s\right)^2 + R^2\frac{Q_2^2}{2} + \frac{P_s^2}{2}\frac{K}{K_x^2 + K} \qquad (33)$$

The type of particle motion with some longitudinal momentum spread isn't changed. The center of ellipse painting the motion of the Larmour circle is shifted on value

$$\Delta X = \frac{K_x}{K_x^2 + K} p_s . \qquad (34)$$

The fast rotation the particle acquires the additional momentum induced by the bending force

$$\frac{\Delta \rho_y}{p_{s0}} = -\frac{K_x}{R}\frac{\Delta \rho_s}{p_{s0}} . \qquad (35)$$

The term

$$\frac{p_s^2}{2}\frac{K}{K_x^2 + K} \qquad (36)$$

describes the "effective mass" of particle in bending.

This result can be obtain easy from the usual equation of motion in the ordinary *(x,y,x',y')* variable. For the particle with momentum derivation $p+\Delta p$ the equation of motion is

$$x'' + k_x^2 x + K y' = \frac{1}{R}\frac{\Delta p}{p}, \quad y'' + k_x^2 y - K x' = 0, \qquad (37)$$

The particle with momentum $p+\Delta p$ placed in the reference orbit is not balanced between Lorentz force and centrifugal force. Therefore, the particle shifts to new orbit

$$x = \frac{1}{R}\frac{\Delta p}{p}\frac{1}{k_x^2} = \frac{R}{1-n}\frac{\Delta p}{p}, \quad y = 0 . \qquad (38)$$

So, the dispersion in storage ring with longitudinal magnet field is the same as in the usual ring with smooth focusing. The longitudinal field is not influence on placement of new orbit. It defines only the character of motion around new orbit. By change the initial reference momentum *p*, the particle begins slow drift oscillations around new equilibrium position. If *n*=0.5, it is circles in transverse plane of beam. With decreasing of *n* the circle degenerates into the ellipse with large axis in vertical direction. The period of oscillation around new equilibrium position is much larger than period of revolution. Therefore, the view and interpretation of one-turn periodic solution for dispersion function can be complicated.

As in any weak-focusing lattice, the negative mass effect is presented in simplest storage ring with longitudinal magnet field. The particle with momentum increment moves back relatively to the particle on reference orbit. The negative effective mass is conditioned on predominance of path length increasing effect over effect of longitudinal velocity increase. The value of relevant longitudinal shift is

$$\Delta\Theta = \frac{v\Delta t}{2\pi R} = -\frac{n}{1-n}\frac{\Delta p}{p} . \qquad (39)$$

Such property of motion can lead to spontaneous modulation of proton density as a result of negative mass instability.

The Figures 1 and 2 show the example of the particle motion in the storage ring with strong longitudinal magnetic field. In the Figure 3, the transverse motion of particle with derivation of momentum is shown.

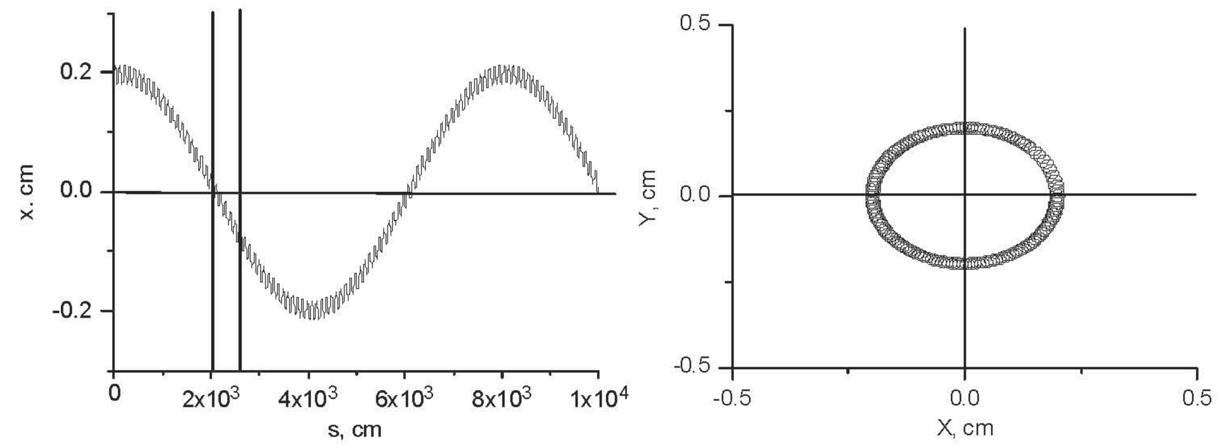

Figure 1. Phase pictures of the particle motion in the storage ring with longitudinal magnetic field *B=20 kG*. The field index is *n=0.5.* The vertical straight lines are pointed the circumference of the storage ring. The particle is proton with energy *2 MeV*. The ring radius is *100 cm*.

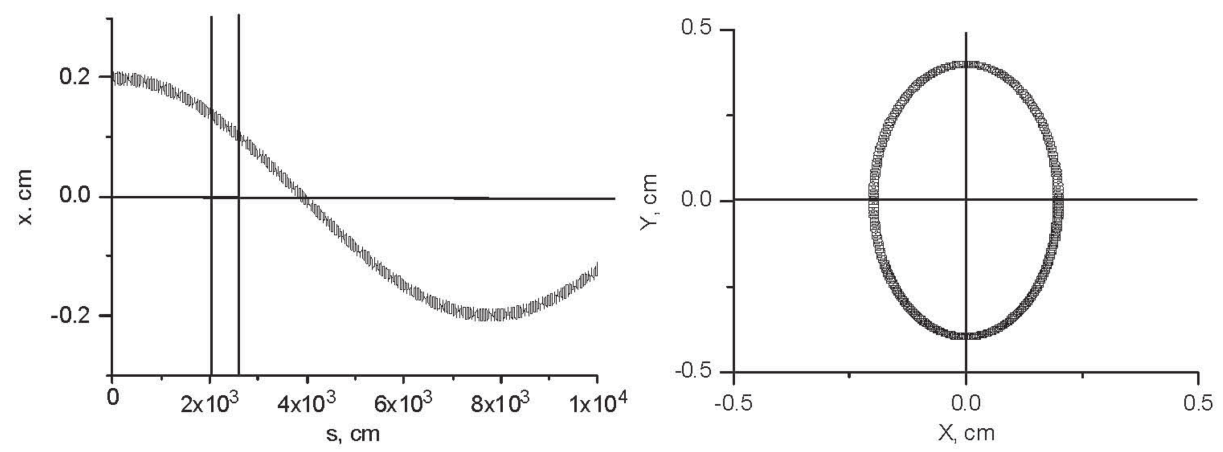

Figure 2. Phase pictures of the particle motion in the storage ring with longitudinal magnetic field *B=20 kG*. The field index is *n=0.2*.

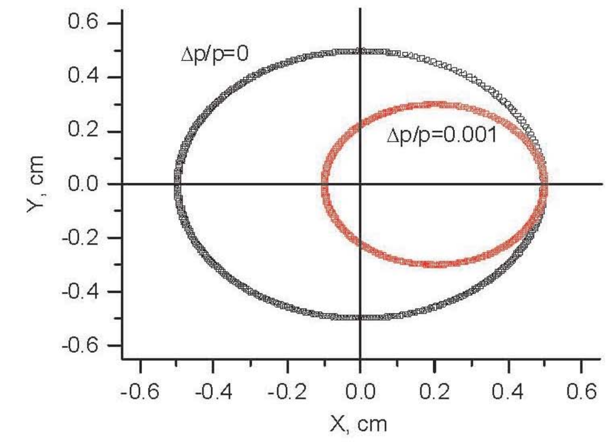

Figure 3. Transverse motion of particle with longitudinal momentum deviation. Field index value is *n*=0.5. Longitudinal magnet field is *B=20 kG*, ring radius is *100 cm*. The initial coordinates are *X=1.0, Y=0, X'=0.003, Y'=0*. The proton energy is *2 MeV*.

Summary

The systems with longitudinal magnetic field are characterized by the strong coupling between the horizontal and vertical motion. In this case it is more natural the new variables for describing particle motion. They relate to the fast rotation of the particle around the magnetic force line and slow drift of center of this rotation. The equations describing these motion modes are weak coupled at the limit of the strong longitudinal magnetic field. In the limit of the infinite magnetic field this motion modes may be considered as uncoupling. But at a low value of the longitudinal magnetic field the coupling is strong. Thus, this situation is opposite to the classical case when the initial uncoupling vertical and horizontal motion is coupled by a weak magnetic field.

References

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