Measurement of SUSY Parameters

Bob McElrath
University of California, Davis

Work(s) in progress with Zhen-Yu Han, Hsin-Chia Cheng, Jack Gunion, and Guido Marandella

Aspen Collider Physics Workshop 2007, Jan 8 2007

Motivation

If our current particle picture of Dark Matter is correct, the LHC is likely to be a Dark Matter factory. Realistic models containing a Dark Matter particle tend to be very similar.

- A symmetry is added to keep Dark Matter stable $\rightarrow$ Dark Matter is produced in pairs.

- Symmetries which keep Dark Matter stable are often taken from other sources (because we prefer as simple a model as possible), such as:

  - Proton Stability (R-Parity in SUSY)

  - Custodial Symmetry (solving Little Hierarchy Problem)

  - 5D momentum conservation (KK number conservation in UED)

  "Other Sources" for the symmetry generically means "Other Particles".

Other Particles

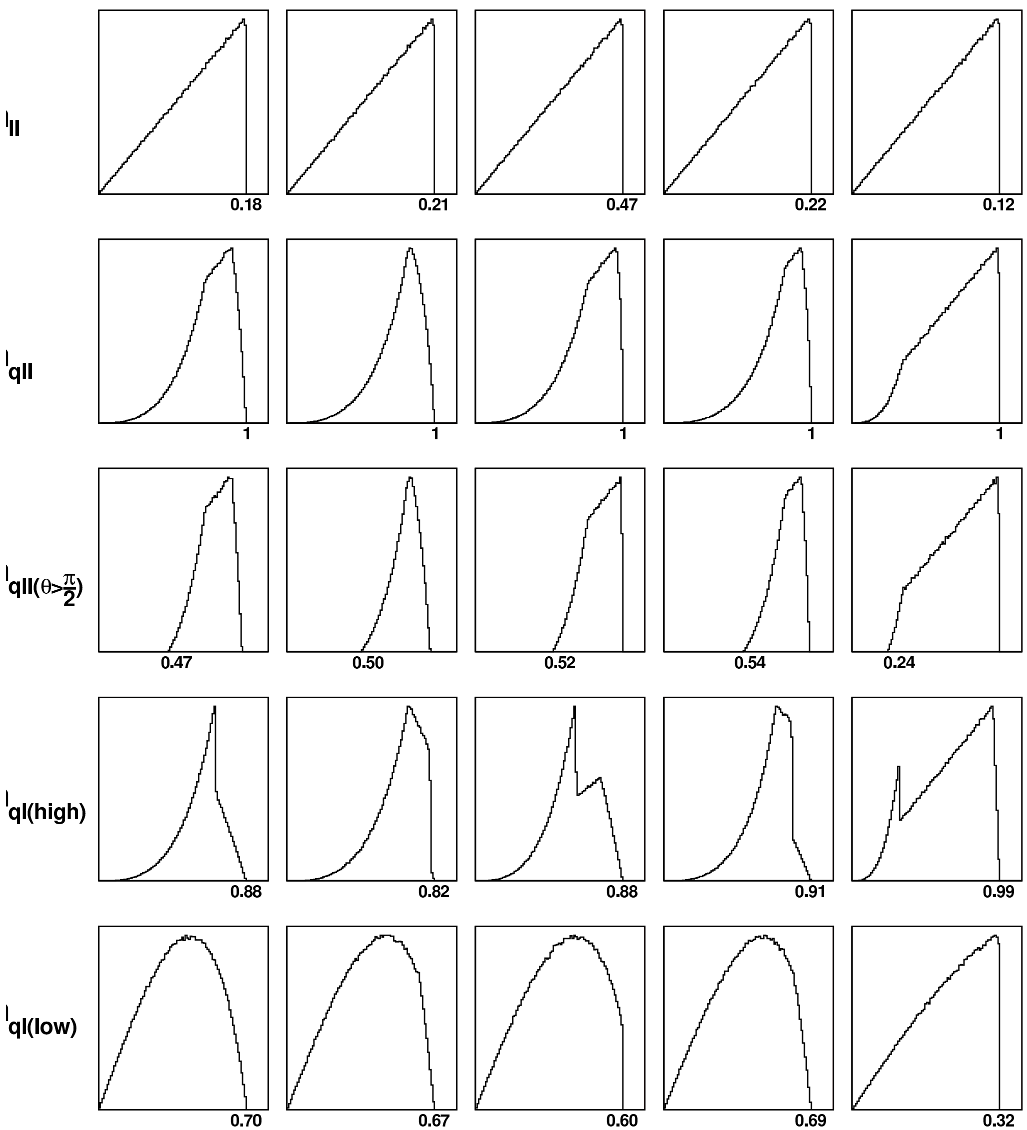
Other Particles means that the Dark Matter is generically produced from the decay of a heavier particle which has SM quantum numbers. (such as a squark, slepton, T-Parity-odd fermion, etc)

All masses require one to add the missing particle to a visible particle. Therefore expected signatures of new physics contain *no way to directly obtain the mass of a particle, from the 4-vectors in the event.*

e.g. In a visible decay $Z \to l^+ l^-$, it is trivial to get a consistent estimator for the Z mass: $m^2 = (p_{l^+} + p_{l^-})^2$. This estimator m^2 has the property that its mean, $\langle m^2 \rangle$ converges to the Lagrangian parameter M_Z^2. This provides the strongest available indicator (to cut on) indicating that this is a Z.

For events with missing particle, no such thing is possible. (yet!)

Existing Studies (Barr Method)



Existing studies all on fitting distribution in a variable correlated to masses. If we must rely on such things, this is troublesome

- Mass determinations are very sensitive to a small number of events (those occuring at inflection points and endpoints).
- Detector resolution makes all (Barr-type) distributions look similar.

[Gjelsten, Miller, Osland `hep-ph/0410303`]

Existing Studies: Cross Sections as Probability Densities

What are these studies doing, from a theoretical/statistics perspective? First let us define a probability distribution for an event. A cross section generally is given by

$$\sigma = \frac{1}{F}\int \left|\mathcal{M}(p_0^\mu, p_i^\mu, \mathbf{Y})\right|^2 \left(\prod_i \frac{d^3\vec{p}_i}{(2\pi)^3 2E_i}\right)(2\pi)^4\delta^4(p_0^\mu - \textstyle\sum_i p_i^\mu)$$

for some initial state momenta p_0^μ and final state momenta p_i^μ. This is a zero-dimensional projection of a high-dimensional phase space, and contains very little information! Buried in here somewhere is all the information that is to be had. Let us do a little rearrangement to retain all information in the high-dimensional space.

$$P(\vec{p}_1, \ldots, \vec{p}_N) = \frac{1}{\sigma}\frac{d\sigma}{\prod_i d^3\vec{p}} = \frac{(2\pi)^{4-3N}}{2^N F\sigma \prod_i E_i}\left|\mathcal{M}(p_0^\mu, p_i^\mu, \mathbf{Y})\right|^2 \delta^4\left(p_0^\mu - \textstyle\sum_i p_i^\mu\right).$$

this is a *probability density* expressing the probability of a particular configuration of momenta. For N external particles, it is a $3N-4$ dimensional space.

Cross Sections as Probability Densities II

$$P(p_i^\mu) = \frac{1}{\sigma}\frac{d\sigma}{\prod_i d^3\vec{p}} = \frac{(2\pi)^{4-3N}}{2^N F\sigma \prod_i E_i}\left|\mathcal{M}(p_0^\mu, p_i^\mu, \mathbf{Y})\right|^2 \delta^4\left(p_0^\mu - \sum_i p_i^\mu\right). \quad (1)$$

In principle, one could directly compare this PDF (*Probability* Density Function) between simulated events and data. But, high-dimensional spaces require a lot of data to map out.

So, let us project this PDF onto a lower dimensional space.

$$P(E_1) \;=\; \int \frac{P((E_1; p_{1x}, p_{1y}, \sqrt{E_1^2 - m_1^2 - p_{1x}^2 - p_{1y}^2}), p_i^\mu)}{2(2\pi)^3 F\sigma\sqrt{E_1^2 - m_1^2 - p_{1x}^2 - p_{1y}^2}} \times dp_{1x}dp_{1y}\prod_{i\neq 1} d^3\vec{p}_i$$

where we have changed variables $p_{1z} = \sqrt{E_1^2 - m_1^2 - p_{1x}^2 - p_{2x}^2}$.

In this way we can obtain the *shape* of any distribution. All one-dimensional variables can be obtained in this manner, by performing an appropriate projection.

Projections Lose Information

Any projection generically loses information, unless it is an optimal estimator for the parameter of interest.

An optimal estimator is one that factorizes. For example in $Z \to l^+ l^-$,

$$P(X|Y) = P(m|M_Z)P(X-m|Y-M_Z) \tag{2}$$

where by $X - m$ and $Y - M$ I mean all other observables except $m^2 = (p_{l+} + p_{l-})^2$ and all other parameters except M_Z.

Factorizations are only approximate in collider physics. (e.g. up to interference effects)

The Neyman-Pearson lemma tells us that the most powerful statistic for differentiating two hypotheses $\mathbf{Y}$ and $\mathbf{Y}^{(n-1)}$ is the ratio of two Likelihoods. Our Likelihood is

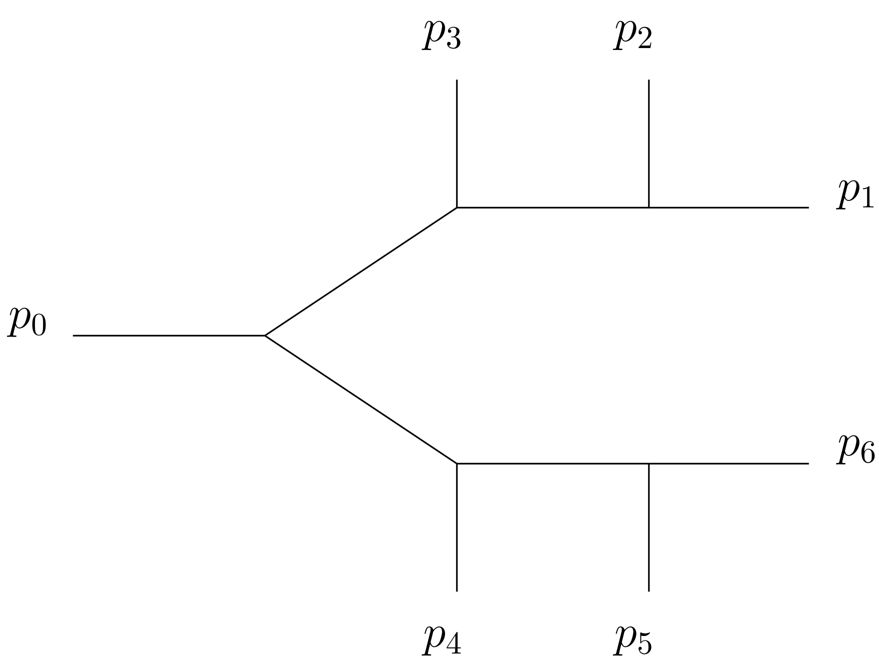
$$L(\mathbf{Y}|\mathbf{X},\mathbf{X}') = \prod_{i=1}^{N} P(\mathbf{x}_i, \mathbf{x}'_i|\mathbf{Y}).$$

Doing this is commonly called "Matrix Element Methods".

Matrix Element Methods are powerful, but also complicated. (We have several projects in progress on this)

Let's try something simpler. What can we do with just phase space? (assuming the Matrix Element is a constant) Our good estimators such as $(p_1 + p_2)^2$ are just phase space!

To examine this, let us choose the $t\bar{t}$ di-lepton topology, which is identical to many interesting SUSY decay topologies.



We have generalized the methods I will describe to any process with exactly 2 missing particles, and 2 or more visible particles.

The vector p_0 is the initial state. This diagram is kinematic. e.g. this all works for t-channel production.

Changing Variables

If we want to talk about masses, the first thing we had better do is change variables.

The $t\bar{t}$ di-lepton topology at the LHC contains 4 kinematic unknowns, which is nice because it also has 4 unknown masses.

$$
\begin{aligned}
a &= (p_2 + p_4 + p_6)^2 \\
b &= (p_2 + p_4)^2 \\
c &= (p_1 + p_3 + p_5)^2 \\
d &= (p_1 + p_3)^2 \\
0 &= \not{p}_x - p_{1x} - p_{2x} \\
0 &= \not{p}_y - p_{1y} - p_{2y} \\
0 &= \sqrt{s}\sigma - p_{vz} - p_{1z} - p_{2z} \\
0 &= \sqrt{s}\tau - E_v - E_1 - E_2 \\
M_1^2 &= E_1^2 - \vec{p}_1^2 \\
M_2^2 &= E_2^2 - \vec{p}_2^2
\end{aligned}
$$

This variable change is non-linear, and incurs a Jacobian J (important if you want to integrate your Probability Density in the mass basis!)

Changing Variables, ctd...

Writing the same thing in integral form, first write the PDF with the Dark Matter's mass constraint explicitly.

$$
\begin{aligned}
P(\mathbf{X}|\mathbf{Y}) \;=\; & f(\mathbf{X},\mathbf{Y}) \int |\mathcal{M}(\mathbf{Y}, p_0^\mu, \ldots, p_N^\mu)|^2 \\
& \times \delta^4(p_0^\mu - \textstyle\sum_i p_i^\mu)\, 2E_1\, \delta(E_1^2 - m_1^2 - |\vec{p}_1|^2)\, 2E_2\, \delta(E_2^2 - m_2^2 - |\vec{p}_2|^2) \\
& \times d\tau\, d\sigma\, d^3\vec{p}_1\, d^3\vec{p}_2\, dE_1\, dE_2
\end{aligned}
$$

Next expand the dimensionality by 4 and add 4 delta functions, corresponding to the 4 propegators.

$$
\begin{aligned}
P(\mathbf{X}|\mathbf{Y}) \;=\; & f(\mathbf{X},\mathbf{Y}) \int |\mathcal{M}(\mathbf{Y}, p_0^\mu, \ldots, p_N^\mu)|^2 \\
& \times \delta^4(p_0^\mu - \textstyle\sum_i p_i^\mu) \\
& \times 2E_1\, \delta(E_1^2 - m_1^2 - |\vec{p}_1|^2)\, 2E_2\, \delta(E_2^2 - m_2^2 - |\vec{p}_2|^2) \\
& \times \delta((p_2 + p_4 + p_6)^2 - a)\, \delta((p_2 + p_4)^2 - b) \\
& \times \delta((p_1 + p_3 + p_5)^2 - c)\, \delta((p_1 + p_3)^2 - d) \\
& \times d\tau\, d\sigma\, d^3\vec{p}_1\, d^3\vec{p}_2\, dE_1\, dE_2\, da\, db\, dc\, dd
\end{aligned}
$$

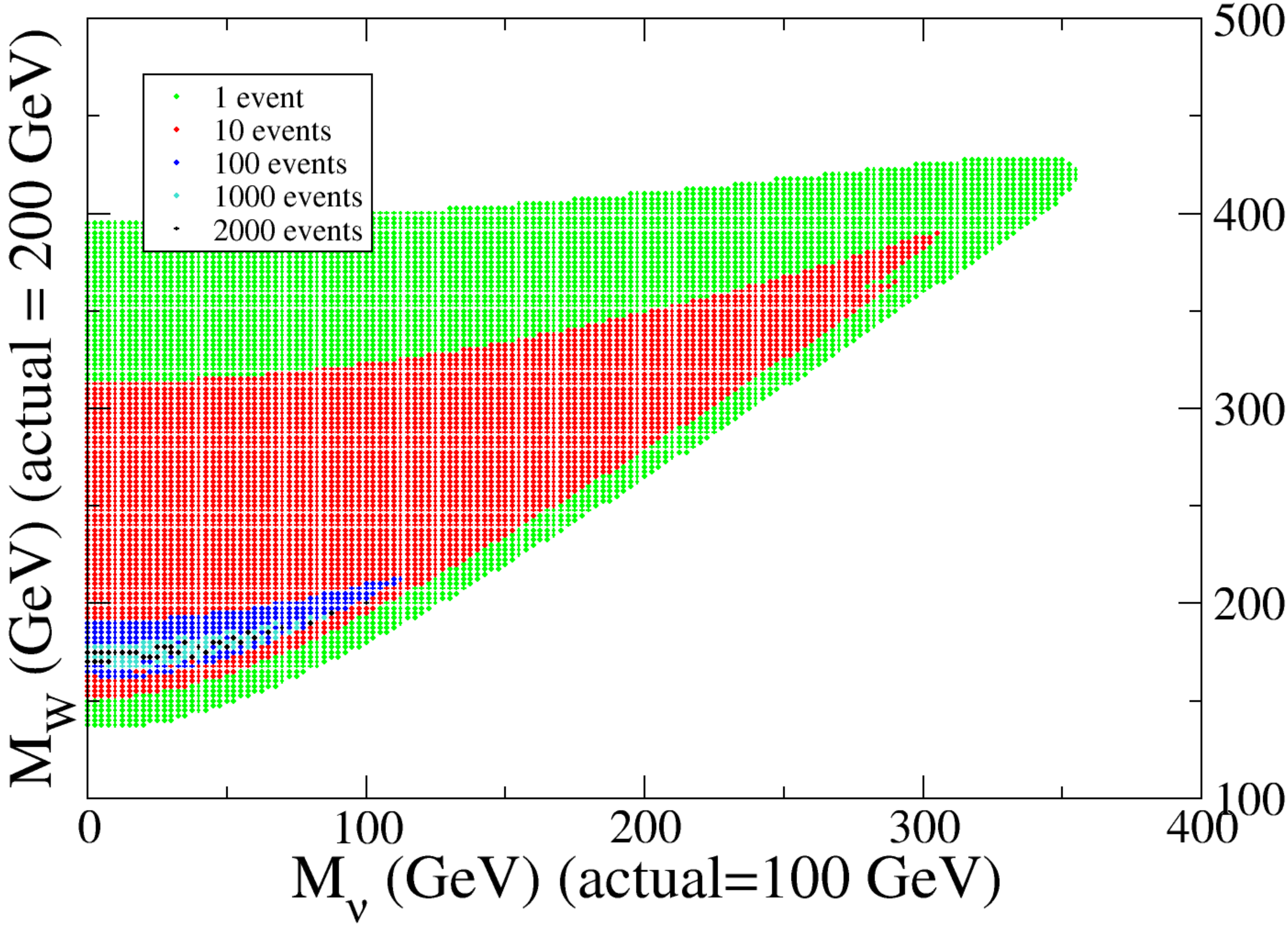
Simplified Likelihood

Let us try to characterize what's going on in mass space, by constructing the likelihood $L(\mathbf{Y}|\mathbf{X}) = \prod_i P_i(\mathbf{X}|\mathbf{Y})$ in the narrow width approximation. Note that our $\delta(a-(p_1+p_3)^2)$ (etc) is exactly what would arise from a Matrix Element containing narrow widths.

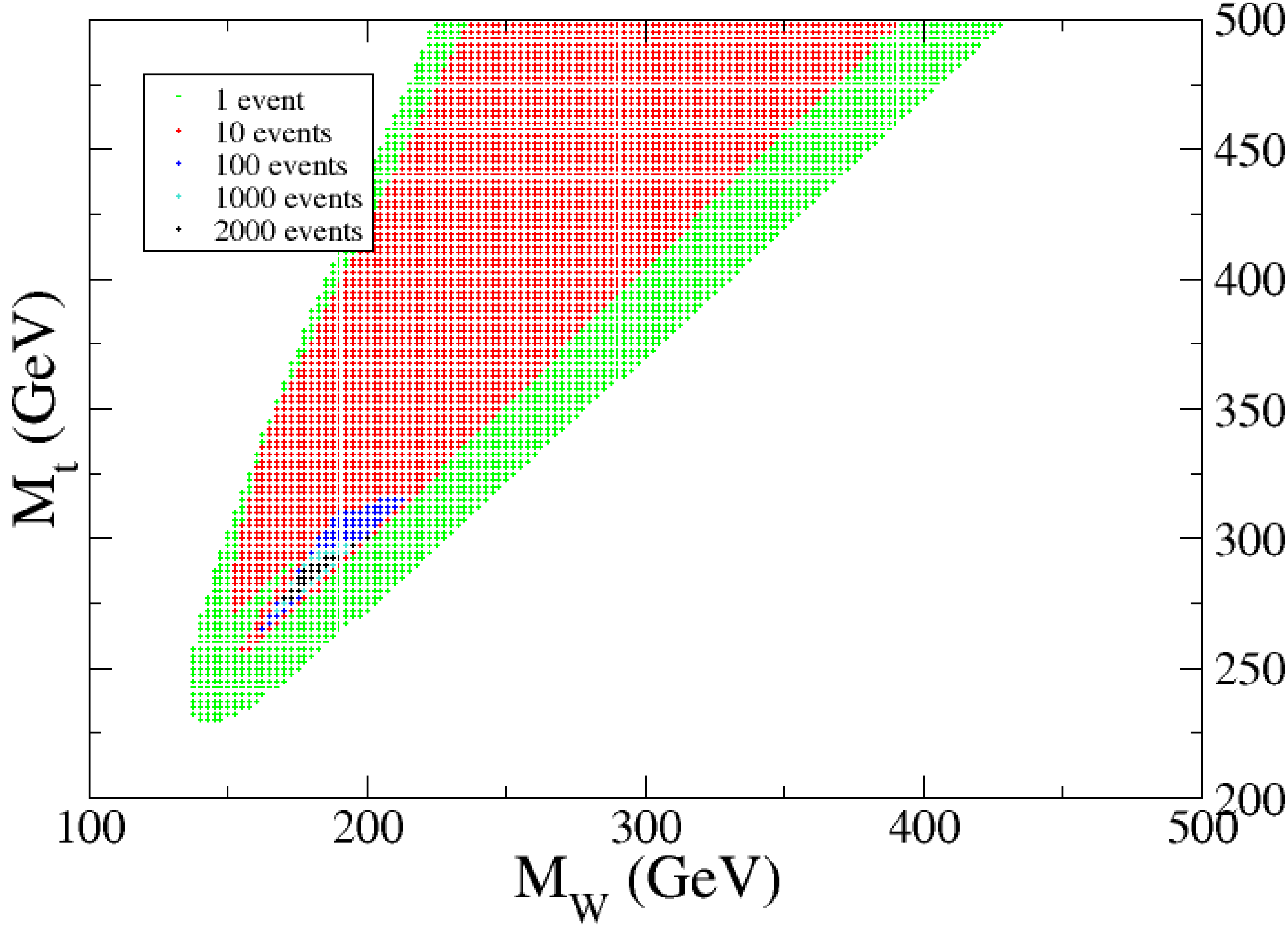
This is identical to taking the integrand to be 1 after our variable change. The $P(\mathbf{X}|\mathbf{Y})$ is zero in regions where the variable change cannot be performed (would result in complex E, p).

It's also equivalent to answering the question: Given two events, *what is the region in mass space that is compatible with both events?*

We extend to N events and ask what region is consistent with *all* events. (no resolution, detector simulation, or combainitorics here)



1 event
10 events
100 events
1000 events
2000 events
0
100
200
300
400
100
200
300
400
500
M_ν (GeV) (actual=100 GeV)
M_W (GeV) (actual = 200 GeV)



1 event
10 events
100 events
1000 events
2000 events
M_t (GeV)
M_W (GeV)
100
200
300
400
500
200
250
300
350
400
450
500

Reading Graph Voodoo

These graphs clearly contain the correct masses.

Now we've got a problem: How do we extract the masses? Most things you might think of are in general biased, inconsistent, or non-convergent.

The remaining region is a *volume*. What do I report as my mass? What are the error bars?

Inconsistent estimator: converges to the wrong value.

Biased estimator: asymmetric convergence. The central value is systematically above or below for low statistics.

Inconsistent and Biased estimators can still be used, and require the sort of tricks we're used to: do a full simulation of the signal for a given set of masses, and wiggle the masses to optimize fits of shapes.

You're beginning to feel dizzy

Let's look in more detail at what's going on.

What we have plotted is the region of mass space *consistent* with an event. This comes from our variable change:

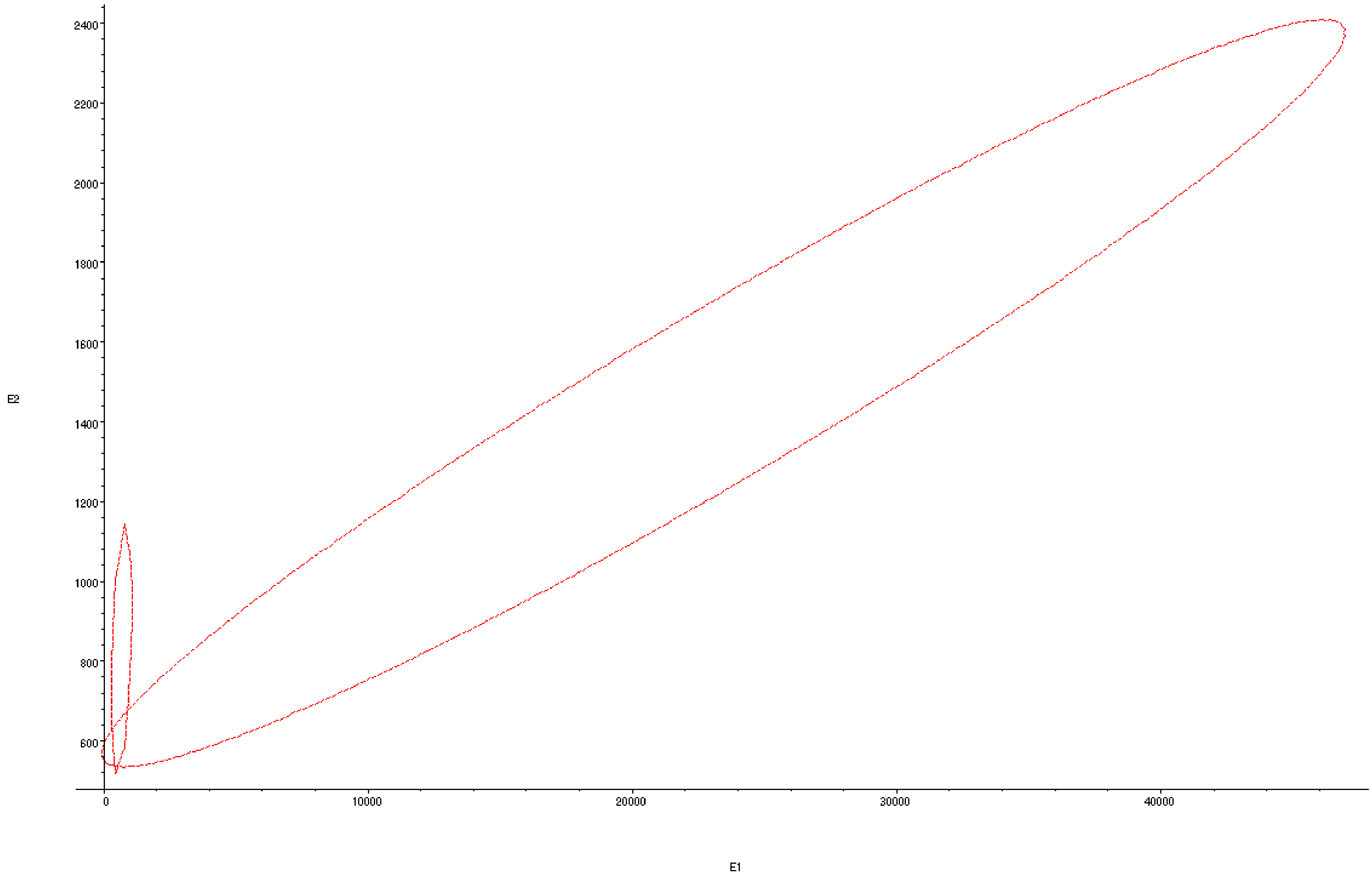
The variable change to mass space results in two quadratic equations.

$$\begin{aligned} A &= a_{11}E_1^2 + a_{12}E_1E_2 + a_{22}E_2^2 + a_1(m)E_1 + a_2(m)E_2 + a_0(m, m^2) \\ B &= b_{11}E_1^2 + b_{12}E_1E_2 + b_{22}E_2^2 + b_1(m)E_1 + b_2(m)E_2 + b_0(m, m^2) \end{aligned}$$

where the mass dependence is as indicated. The preceeding graphs have $P(\mathbf{X}|\mathbf{Y}) = 1$ when this system has a solution, $P(\mathbf{X}|\mathbf{Y}) = 0$ when it does not.

These are two equations of *ellipses* of fixed eccentricity. Moving around in mass space corresponds to translating and scaling the size of the ellipses.

J. Random Event



You're getting dizzier...

These ellipses can also be written using the vector $x_i = (E_1, E_2, 1)$ as

$$A = x_i f^{ij} x_j = 0, \qquad B = x_i g^{ij} x_j$$

using the tensors

$$f^{ij} = \begin{bmatrix} a_{11} & a_{12} & a_1(m) \\ a_{12} & a_{22} & a_2(m) \\ a_1(m) & a_2(m) & a_0(m, m^2) \end{bmatrix} \qquad g^{ij} = \begin{bmatrix} b_{11} & b_{12} & b_1(m) \\ b_{12} & b_{22} & b_2(m) \\ b_1(m) & b_2(m) & b_0(m, m^2) \end{bmatrix}$$

For determining masses, we only care about *whether* a solution exists, not the actual values of E_1, E_2. Therefore, without loss of generality, we are free to rotate, translate, and scale the vector x_i.

$$f^{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -r^2(m, m^2) \end{bmatrix}, \quad \begin{array}{c} x^2 + y^2 - r^2(m, m^2) = 0 \\ \frac{(x - x_0(m))^2}{a^2} + \frac{(y - y_0(m))^2}{b^2} - R^2(m, m^2) = 0 \end{array}$$

$$g^{ij} = \begin{bmatrix} 1/a^2 & 0 & -x_0(m)/a^2 \\ 0 & 1/b^2 & -y_0(m)/b^2 \\ -x_0(m)/a^2 & -y_0(m)/b^2 & -R^2(m, m^2) + x_0(m)^2/a^2 + y_0(m)^2/b^2 \end{bmatrix}$$

The room is spinning

The surface of mass is defined by where these two ellipses touch at a single point. This is an 8^{th} order polynomial in masses, and encapsulates *all information and correlations in the event*.

How do we get a mass out of this?

If the region were finite, we could use the centroid

$$\langle m \rangle = \frac{\int m S(m) dm}{\int S(m) dm}$$

but this is not defined because the volume is infinite.

Another way to find the centroid is to find the extrema, solving the system

$$S(m_1, m_2) = 0, \qquad \frac{\partial}{\partial m_1} S(m_1, m_2) = 0$$

The Dizzy Mass

The **Dizzy Mass** is:

The mass defined by the centroid of the region in mass space that is kinematically compatible with the event

This method produces an estimator for each mass in the event (including the Dark Matter!) *for every event*!

One can simply put this value in a histogram, fit it, cut it, slice it and dice it.

We will distribute code to compute this.

Golden Events

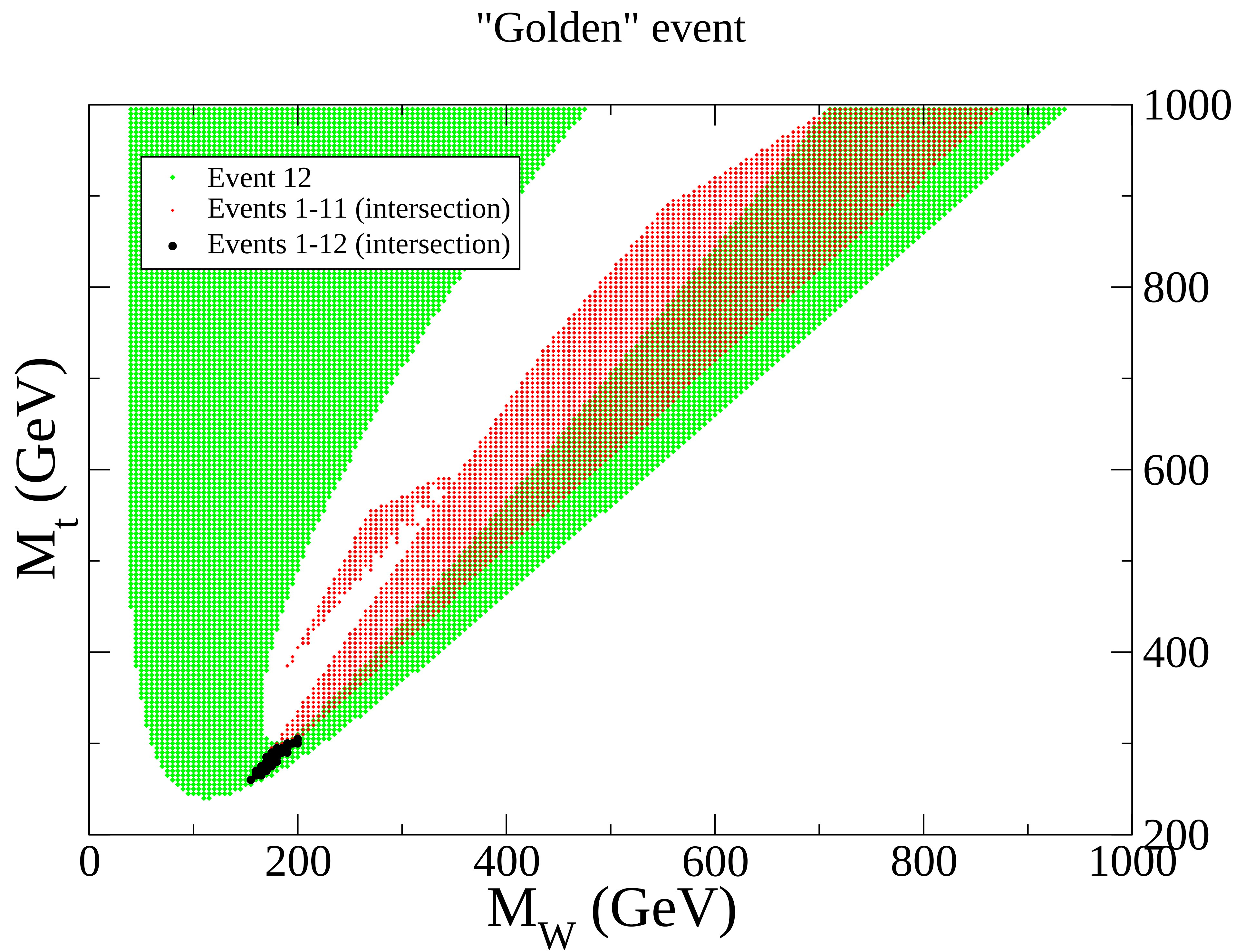
Some events have the property that one Energy ellipse can fit *inside* the other. This means there is a hole in the allowed mass region.

We call these "Golden Events" because when intersected with a non-golden event, *very little volume remains*.

It is these events that cut off the infinite-mass solutions when intersected.

All events have this inner surface, but for most events it corresponds to going from 2 solutions for (E_1, E_2) to 4 solutions. (e.g. the ellipse is to large to fit inside the circle)

Therefore it is the *inner* mass surface that provides good estimators. The *outer* surface always allows infinite M_t and $M_\nu = 0$.



"Golden" event
Event 12
Events 1-11 (intersection)
Events 1-12 (intersection)
M_t (GeV)
M_W (GeV)
0
200
400
600
800
1000

Summary/Conclusions

Existing techniques (shapes/edges) are in general biased, inconsistent, and very sensitive to a small number of events.

We present a new estimator for masses in events with missing energy, the **Dizzy Mass** defined by the centroid of the region in mass space that is kinematically compatible with an event.

We will provide a C++ code to compute the Dizzy Mass for use by everyone.

A proper estimator means: Statistical errors scale properly with the number of events.