Fermilab

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Scaling for Faster Macroparticle Modeling of Accelerator Multiparticle Longitudinal Dynamics

- Macroparticle tracking is a direct way to follow the evolution of a longitudinal phase space distribution.

- Many macroparticles are required when the particles interact through short range wake fields or the inter-particle force is included.

- A simple scaling has been found which reduces both the number of macroparticles required and the number of tracking steps per unit simulated time.

Main Points

- Concept
- Invariance of Vlasov equation under scaling
- Calculation time $\propto$ (scaling constant)$^{-4}$
- Comparison of scaled with un-scaled cases
- Limitations

Introduction

Multiparticle tracking has established utility for modeling evolution of longitudinal phase space distributions of particles in accelerators as they respond to the rf in acceleration or bunch manipulation. One goes from single particle to multiparticle dynamics by calculating the beam current every time step and including its effect on the single particle motion. However, the number of macroparticles needed and bandwidth required for quantities in the frequency domain need careful attention. It is very easy to generate spectacular spurious instabilities by long time steps or too few macroparticles.

Heretofore boggling calculations can be scaled to the desktop and normal patience. Recent studies of high brightness injectors and so-called factory accelerators give these considerations current importance.

Equations of Motion

In a multiparticle tracking calculation, the positions of representative macroparticles in an energy-phase plane are found by iterating a pair of first order difference equations. The iteration step is usually one period of circulation of particles around the ring; it can be fractional or multiple turns. The collective forces are calculated each iteration from the fields generated by the total beam current.

A suitable single turn map is [1]

$$\begin{aligned} \varphi_{i,m} &= \varphi_{i,m-1} + \frac{2\pi h\eta}{\beta_s^2 E_s}\varepsilon_{i,m-1} \\ \varepsilon_{i,m} &= \varepsilon_{i,m-1} + eV(\varphi_{i,m} + \phi_{s,m}) - eV(\phi_{s,m}) \end{aligned} \quad ,$$

where φ is a phase difference between a particle and the synchronous phase ϕ_s, likewise ε is the energy difference between a particle and the synchronous energy E_s, i labels particles, and m labels turns.

Scaling Concept

It appears that the phase space motion can be accelerated by scaling the phase slip factor η and the potential up by the same factor. The potential is not necessarily just that from the rf system; the collective potential enters identically.

The next slides compare the collective potential *vs*. rf phase for an h=1 bunch with $4 \cdot 10^{13}$ protons in a calculation scaled by 2 with an un-scaled modeling. The storage ring parameters used are similar to those of the Los Alamos PSR with the addition of a resonance (HOM) 1% above the third harmonic of the the beam circulation frequency $f_\circ$; it is the only source of collective potential included.

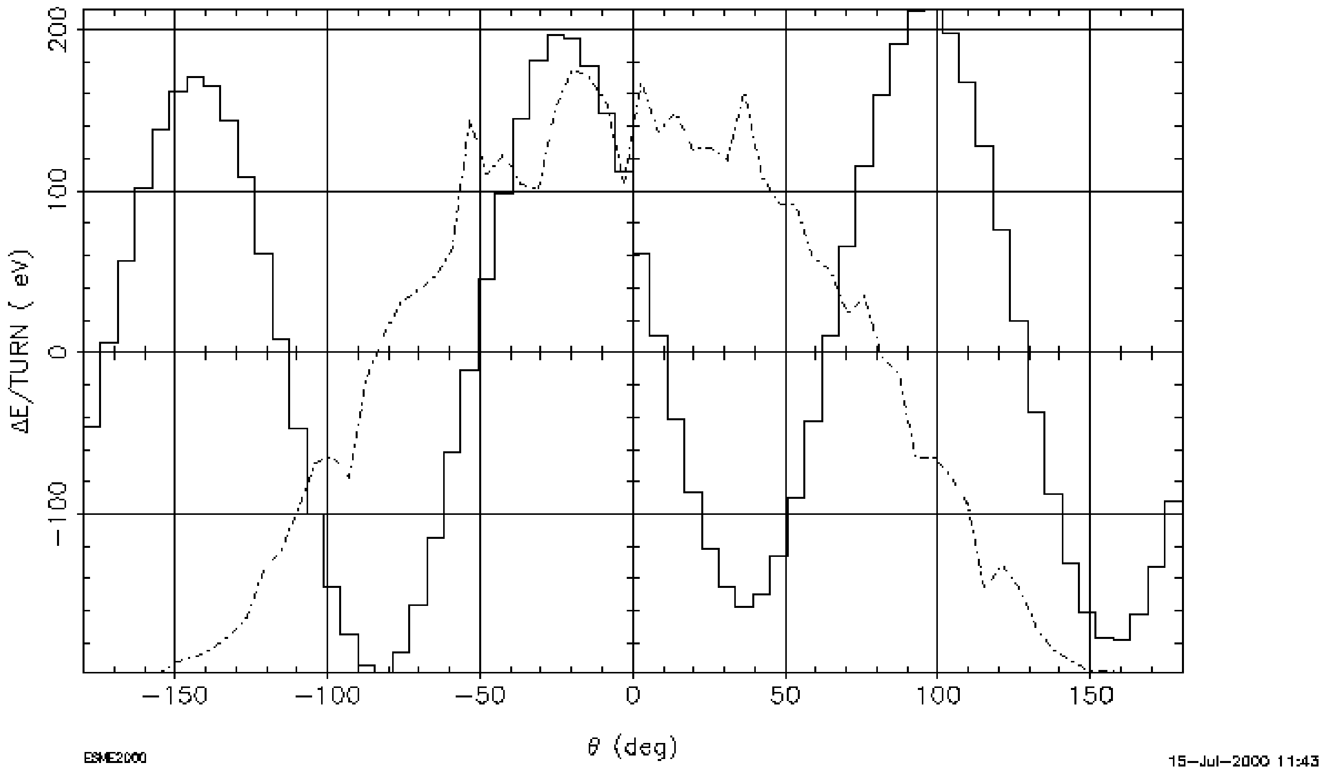
Table I: **Storage ring parameters used in tracking**

Circumference	C	90.261	m
momentum	p	1459.7	MeV/c
transition energy/$m_\circ c^2$	γ_T	3.08	
slip factor $\gamma_T^{-2} - \gamma^{-2}$	η	.18694	
rf peak voltage	V_{rf}	5	kV
rf harmonic	h	1	
beam circulation frequency	$f_\circ$	2.7940	MHz
HOM frequency	f_{res}	8.4658	MHz
HOM impedance at resonance	R_{sh}	300	Ohm
HOM Q	Q	100	

Table II: **Definition of frequently used symbols**

ϕ	rf phase
ϕ_s	synchronous phase
φ_i	difference between particle phase and ϕ_s
i, j, or k	index for particles
m	index for turns
h	rf harmonic number
e	elementary particle charge (> 0)
f	frequency
E_s	synchronous energy
β	relativistic velocity v/c
γ	relativistic energy $E_s/m_\circ c^2$
γ_T	γ of transition energy in synchrotron
η	phase slip factor $(\gamma_T^{-2} - \gamma^{-2})$
V	total potential
V_{rf}	peak rf voltage
V_{coll}	collective potential
λ	dimensionless real scaling constant
Z	longitudinal impedance
$\omega_\circ$	angular frequency of beam circulation
ω_s	small amplitude oscillation angular frequency
Q_s	synchrotron tune
ε_j	energy of j-th particle minus E_s
τ_j	time difference $\varphi_j/h\omega_\circ$
p	index for fourier harmonics
$\hat{\tau}_k$	time amplitude in synchrotron oscillation
ψ_k	synchrotron oscillation phase of a particle
N	number of particles in beam
Ψ	particle density in phase space

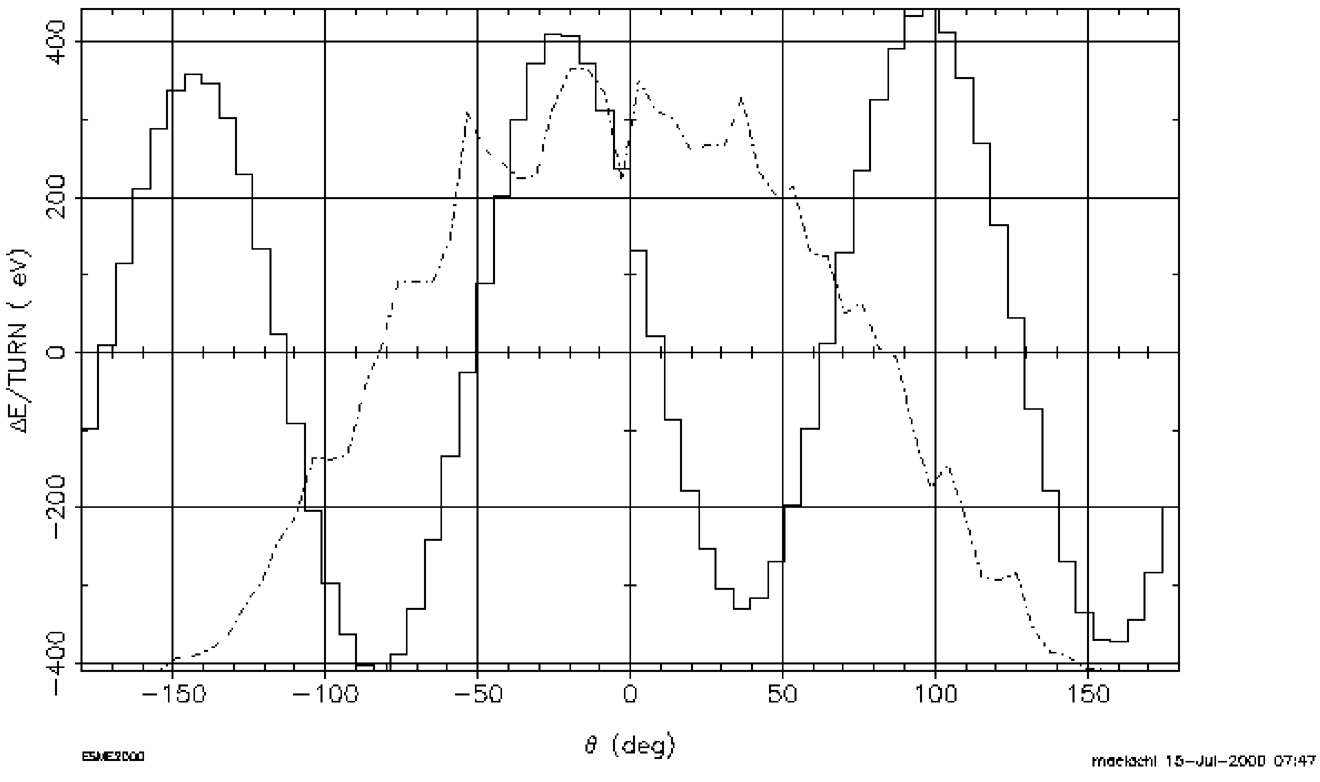
Voltage *vs.* rf phase (histogram) – Not Scaled



After $4 \cdot 10^4$ turns (0.014 s)

There is an h=3 HOM resonance in the ring whose parameters are given in Table II. The beam intensity is $4 \cdot 10^{13}$ protons. The longitudinal charge distribution is shown with the same binning by the (— ·) curve.

As Before – Scaled



Conditions same as preceding except scaled with $\lambda = 2$ after $2 \cdot 10^4$ iterations. The voltage scale is $V' = \lambda V$.

- The particle distributions are identical when the plot for time t in the un-scaled calculation is compared to time t/λ in the scaled calculation.

- Note the factor of two scale difference on the potential axis in the two plots.

- The scaled charge distribution looks slightly smoother because of faster mixing between adjacent bins in the scaled case.

- Obvious gain is a factor λ^{-1} in the computing time by speeding up the clock in the scaled calculation.

- Scaling up of the time means that frequencies like the rf frequency and resonance frequencies in $Z_{||}$ must also be scaled.

- With broadband impedance or direct interparticle forces consequences of frequency scaling are different and very advantageous.

- If beam pipe impedance must be covered over some particular frequency bandwidth (fields down to some minimum range) it is necessary to bin the charge distribution in bins narrow enough to reflect detail to that scale.

- Fourier components of the beam current must span bandwidth.

- For beam circulation frequency $f_\circ$ and bandwidth W, the charge is divided into $2W/f_\circ$ bins for the finite transform.

- In the scaled system $f_\circ$ is λ times higher; the number of bins can be reduced by a factor λ^{-1}.

- Number of harmonics is reduced, but frequency range covered is the same.

- Sensitivity lost to features on the scale of $f_\circ$ in the frequency dependence of $Z_\|$.

- Space charge term in frequency domain is

$$\frac{Z_{\text{SC}}}{n} = -i\frac{Z_{\circ} g}{2\beta\gamma^2}$$

  - n is the harmonic number
  - $Z_{\circ} = \sqrt{\mu_{\circ}/\epsilon_{\circ}} = 377\Omega$
  - $\gamma = E/m_{\circ}c^2$
  - g is the geometric factor for the beam tube.

- Evaluation not compromised by more widely spaced frequency sampling

- Number of macroparticles can be scaled by λ^{-3} with same level of numerical noise
  - shown rigorously for space charge[2]
  - heuristic frequency domain argument for any smooth $Z_{||}(\omega)$[3]

- Thus gain from scaling λ^4 in most cases

Invariance of Vlasov Equation Under Scaling

In a bunch for which the small amplitude synchrotron oscillation frequency $\omega_s = \omega_\circ Q_s$, the steady state collective voltage experienced by a particle with phase ϕ is

$$\begin{aligned} V_{\mathrm{coll}}(\phi) \;=\; & \frac{e\omega_\circ}{2\pi}\sum_{k=1}^{N}{\sum_{p=-\infty}^{\infty}}' e^{ip\phi/h}\{Z(p\omega_\circ) - \\ & \frac{i\omega_\circ\hat{\tau}_k}{2}[(p+Q_s)Z(p\omega_\circ+\omega_s)e^{i(\psi_k+Q_s\phi/h)} + \\ & (p-Q_s)Z(p\omega_\circ-\omega_s)e^{-i(\psi_k+Q_s\phi/h)}]\} \quad , \end{aligned}$$

where the sum over k is taken over all particles. The single particle equations of motion in canonical energy-time variables are

$$\begin{aligned} \frac{\mathrm{d}\varepsilon_j}{\mathrm{d}t} &= \frac{he\omega_\circ^2}{2\pi}[V_{\mathrm{rf}}\cos\phi_s + V'_{\mathrm{coll}}(\phi_s)]\tau_j \\ \frac{\mathrm{d}\tau_j}{\mathrm{d}t} &= -\frac{\eta}{\beta^2 E_\circ}\varepsilon_j \quad . \end{aligned}$$

The rf potential is sinusoidal with amplitude V_{rf}. The energy increment $eV_{\mathrm{rf}}\sin\phi_s$ keeps a particle with phase ϕ_s synchronous as the guide field is changed or energy is lost to the real part of the longitudinal impedance.

The total hamiltonian is

$$H = -\sum_{k=1}^{N}\{\frac{\eta}{2\beta^2 E_\circ}\varepsilon_k^2 + \frac{eh\omega_\circ^2}{4\pi}[V_{\mathrm{rf}}\cos\phi_s + V'_{\mathrm{coll}}(\phi_s)]\tau_k^2\} \ .$$

The time evolution of the distribution is described by the Vlasov equation

$$\frac{\mathrm{d}\Psi}{\mathrm{d}t} = \frac{\partial\Psi}{\partial t} + \frac{\partial\Psi}{\partial\tau}\frac{\mathrm{d}\tau}{\mathrm{d}t} + \frac{\partial\Psi}{\partial\varepsilon}\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} = 0 \ ,$$

where the particle distribution function $\Psi(\varepsilon,\tau;t)$ depends on t only implicitly through ε and τ. The equation expresses Liouville's theorem on conservation phase space density and is valid in the absence of diffusion or external damping of the motion.

The *ad hoc* scaling introduced into the map creates an apparently new physical system, denoted as the primed system, related to the original system by

$$\begin{aligned} \eta' &= \lambda\eta \\ V' &= \lambda V \\ t' &= t/\lambda \ . \end{aligned}$$

If the two systems are physically equivalent, then $\Psi'|_{\lambda t'} = \Psi|_t$.

By direct substitution one finds

$$H' = \lambda H \quad .$$

The Vlasov equation in the primed system is

$$\frac{\partial \Psi'}{\partial t'} + \frac{\partial \Psi'}{\partial \tau'}\frac{\mathrm{d}\tau'}{\mathrm{d}t'} + \frac{\partial \Psi'}{\partial \varepsilon'}\frac{\mathrm{d}\varepsilon'}{\mathrm{d}t'} = 0 \quad ,$$

which leads immediately to

$$\lambda\frac{\partial \Psi}{\partial t} + \lambda\frac{\partial \Psi}{\partial \tau}\frac{\mathrm{d}\tau}{\mathrm{d}t} + \lambda\frac{\partial \Psi}{\partial \varepsilon}\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} = 0 \quad .$$

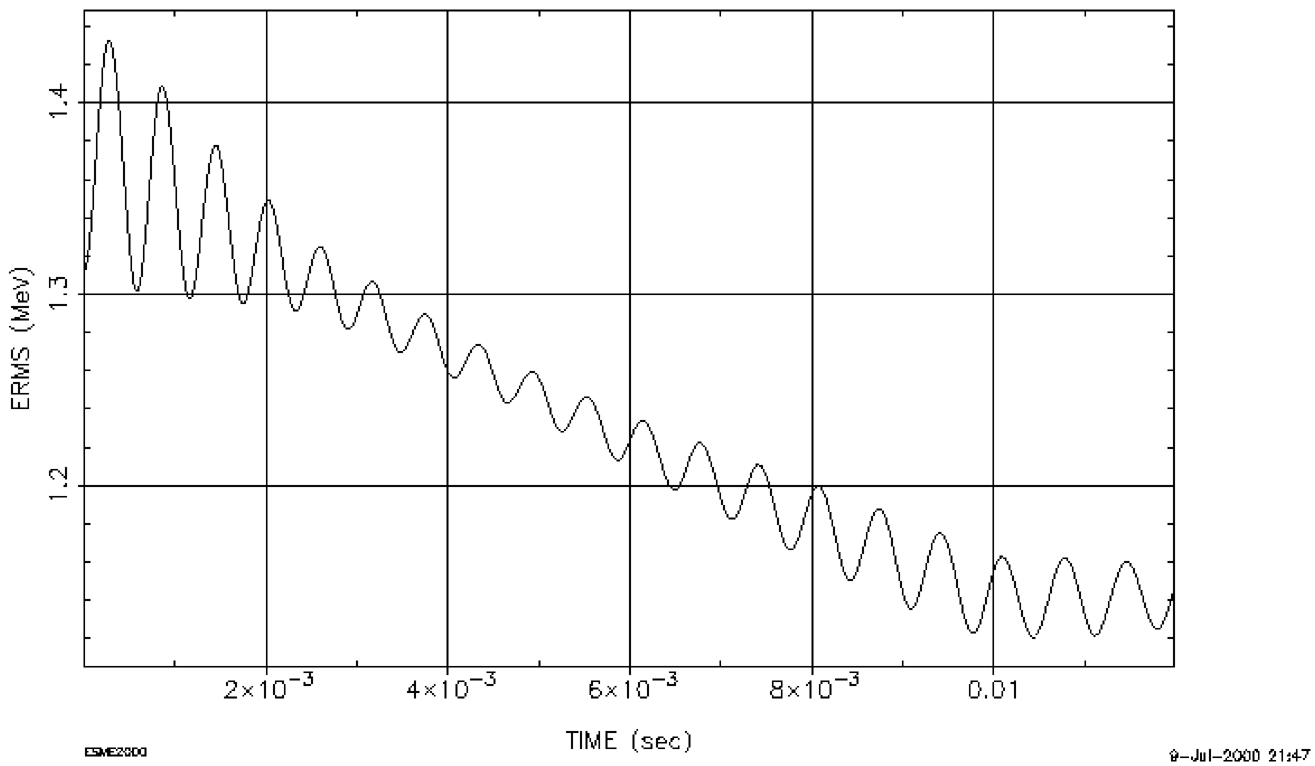
Because the scaling constant $\lambda \neq 0$ appears to the same power in every term of the equation, the solution is independent of the scaling.

Another Comparison of Scaled and Un-scaled Tracking

A more realistic case is to add the space charge force to the previous example. Previously, 8000 macroparticles were quite sufficient because the HOM resonance is at a rather low frequency. When the space charge force is evaluated, the gradient of the charge distribution is calculated; there must be enough macroparticles to produce a smooth distribution. An insufficient number results in large local fluctuation of the force and spurious breakup of the beam into small clumps.

The next two slides show evolution of the rms energy spread of the beam as the charge is raised from zero to a final value of $2 \cdot 10^{13}$ protons over .012 seconds. The macroparticle number was selected arbitrarily and the final charge was adjusted to avoid spurious breakup. The second plot was generated from tracking with the scaling $\lambda = 2$ and ran just a little better than λ^4 times faster. It may not always be possible to scale this much, although in some cases even more may be possible.

RMS Energy Spread in Storage Ring – Un-scaled

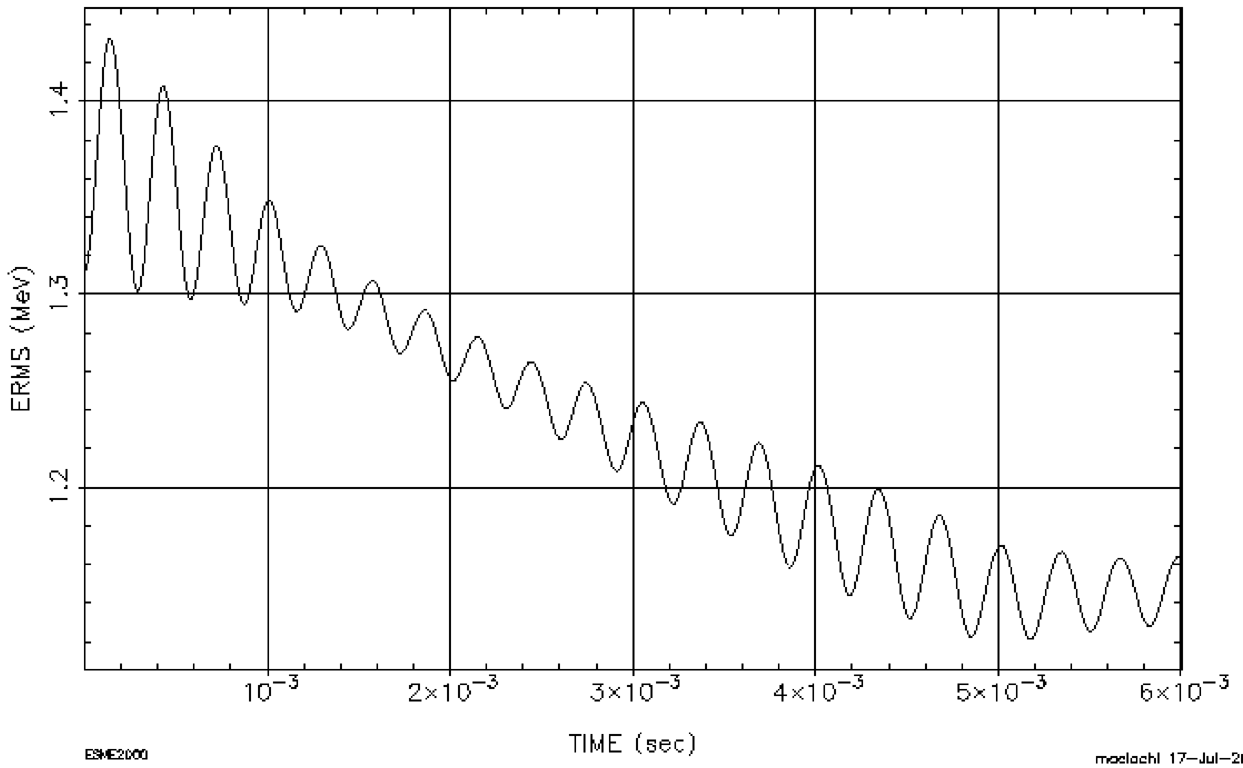


The rms energy spread *vs.* time in the ring specified in Table II including both an h=3 HOM and the perfectly conducting wall space charge force for a beam intensity ramped linearly from zero to $2 \cdot 10^{13}$ protons

- $2 \cdot 10^6$ macroparticles

- 32 hours processor time on Sun Ultra 2

As Before – Scaled $\lambda = 2$



The same conditions used just previously except scaled with $\lambda = 2$ Note that the time scale is $t' = t/\lambda$, one half of that in the previous slide.

- $2.5 \cdot 10^5$ macroparticles
- 1.9 hours processor time on Sun Ultra 2

Utility

- Very effective (λ^4) with space charge and smoothly varying impedance
- Less effective (λ^1) with dominant narrow resonances
- Limitation on λ by maximum allowable spacing of circulation harmonics
- Limitation on λ by acceptable step size
  - $\lambda > 1$ increases effective step size
  - steps in t' must be small fraction of τ_s
  - steps in t' must be small fraction of bunch length and bunch height

- The scaling described is so simple to implement and so innocuous in typical applications that it seems reasonable to employ it when calculation time for the un-scaled case is more than a few minutes. The granularity of the sampling of $Z_{||}(\omega)$ and the adjustment of the frequency of any narrow resonances must be taken into account, but a modeling code can do most of the work.

- I have used mostly frequency domain description for simplicity. The examples, however, are pure time domain calculations using a kludged version of the ESME tracking code.[4]

- A version called ESME2001 now incorporates λ scaling as an option; it appears compatible with all features in both frequency and time domain.

Acknowledgements

The scaling described above has some similarity to that developed by Andre Gerasimov for Fokker-Planck models of stochastic beam cooling,[5] and this work was encouraged by earlier success in using Gerasimov's scaling rules in beam cooling simulations. I received valuable assistance from Zaira Nazario, an undergraduate summer intern from the University of Puerto Rico at Rio Piedras.

References

1. J. MacLachlan, "Difference Equations for Longitudinal Motion in a Synchrotron", Fermilab note FN-529 (1989)

2. J. Wei, "Longitudinal Dynamics of the Non-Adiabatic Regime on Alternating Gradient Synchrotrons", PhD dissertation for State University of New York at Stony Brook (1990)

3. J. A. MacLachlan, "Particle Tracking in E-φ Space for Synchrotron Design and Diagnosis", FERMILAB-Conf-92/333 (1992)

4. The current documentation, containing references to the underlying principles, is most accessible on the ESME web page `www-ap.fnal.gov/ESME`

5. A. Gerasimov, "Macroparticle Simulation for Stochastic Cooling", FERMILAB-Pub-94/376 (1994)