# Tensor reduction and master integrals of the two-loop crossed box

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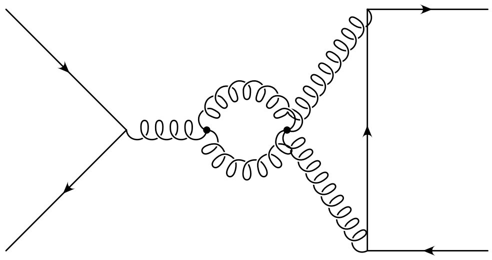
- Scattering processes: $2 \to 2$
  - Tensor integrals
  - Scalar integrals
- Master integrals
- Conclusions and first results

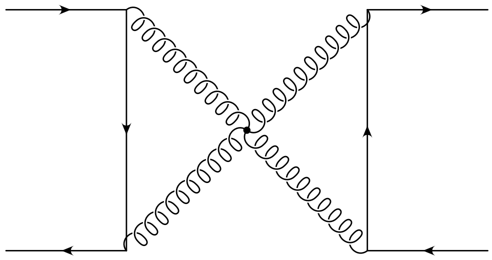
# $2 \to 2$ scattering processes

| Process | Tree | One loop | Two loops |
|---|---|---|---|
| $q\bar{q} \to q'\bar{q}'$ | 1 | 9 | 189 |
| $q\bar{q} \to q\bar{q}$ | 2 | 18 | 378 |
| $q\bar{q} \to gg$ | 3 | 33 | 725 |
| $gg \to gg$ | 4 | 102 | 2473 |

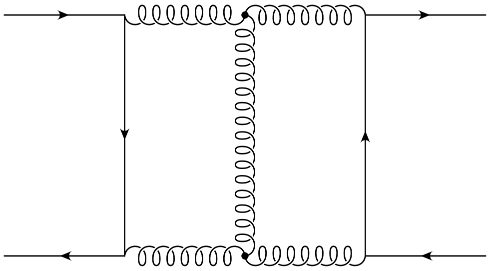
Self energy insertions such as

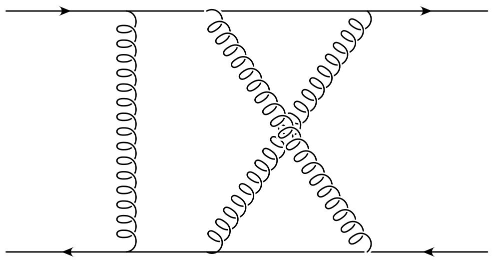


Products of one-loop graphs such as



Genuine new topologies such as planar and crossed double box





# Two main issues

1. Tensor integrals

$$I^D\Big|^{\mu\nu\ldots} = \int \frac{d^D k}{i\pi^{D/2}} \int \frac{d^D l}{i\pi^{D/2}} \, \frac{k^\mu l^\nu \ldots}{A_1 \ldots A_n}$$

2. Scalar integrals and their analytic expansion in $\epsilon$, where $D = 4 - 2\epsilon$.

$$I^D\left(\nu_1 \ldots \nu_n\right) = \int \frac{d^D k}{i\pi^{D/2}} \int \frac{d^D l}{i\pi^{D/2}} \, \frac{1}{A_1^{\nu_1} \ldots A_n^{\nu_n}}$$

# Schwinger parameterization

We start with the generic scalar Feynman integral

$$I^D(\nu_1 \ldots \nu_n) = \int \frac{d^D k}{i\pi^{D/2}} \int \frac{d^D l}{i\pi^{D/2}} \, \frac{1}{A_1^{\nu_1} \ldots A_n^{\nu_n}}$$

Each propagator $i$ can be rewritten using the Schwinger parameter $x_i$

$$\frac{1}{A_i^{\nu_i}} = \frac{(-1)^{\nu_i}}{\Gamma(\nu_i)} \int_0^\infty dx_i \, x_i^{\nu_i - 1} \; \exp(x_i A_i),$$

so that

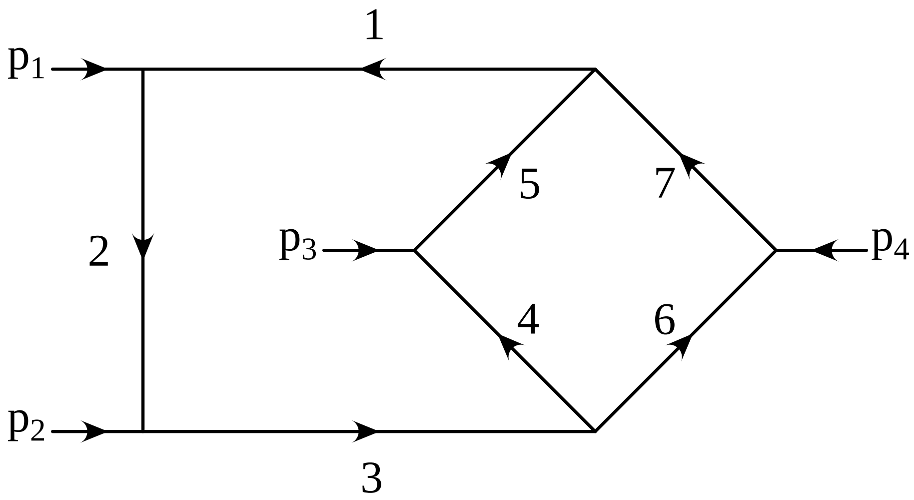
$$\begin{aligned} I^D(\nu_1 \ldots \nu_n) \quad &= \quad \int \frac{d^D k}{i\pi^{D/2}} \int \frac{d^D l}{i\pi^{D/2}} \left( \prod_{i=1}^{n} \frac{(-1)^{\nu_i}}{\Gamma(\nu_i)} \int_0^\infty dx_i \, x_i^{\nu_i - 1} \right) \\ &\quad \times \exp\left( \sum_{i=1}^{n} x_i A_i \right) \end{aligned}$$

where

$$\sum_{i=1}^{n} x_i A_i = a\, k^2 + b\, l^2 + 2\, c\, k \cdot l + 2\, d \cdot k + 2\, e \cdot l + f,$$

and $a$, $b$, $c$, $d^\mu$, $e^\mu$ and $f$ are directly readable from the actual graph, and they are linear functions of the $x_i$.

# The generic two-loop crossed box




$$p_i^2 = 0 \qquad (p_1+p_2)^2 \equiv p_{12} = s \qquad (p_2+p_3)^2 \equiv p_{23} = t$$

$$\begin{aligned}
A_1 &= (k+l+p_{34})^2 &&= k^2 + l^2 + s + 2\,k\cdot l + 2\,k\cdot p_{34} + 2\,l\cdot p_{34} \\
A_2 &= (k+l+p_{134})^2 &&= k^2 + l^2 + 2\,k\cdot l + 2\,k\cdot p_{134} + 2\,l\cdot p_{134} \\
A_3 &= (k+l)^2 &&= k^2 + 2\,k\cdot l + l^2 \\
A_4 &= l^2 &&= l^2 \\
A_5 &= (l+p_3)^2 &&= l^2 + 2\,l\cdot p_3 \\
A_6 &= k^2 &&= k^2 \\
A_7 &= (k+p_4)^2 &&= k^2 + 2\,k\cdot p_4
\end{aligned}$$

$$\begin{aligned}
a &= x_1 + x_2 + x_3 + x_6 + x_7 \\
b &= x_1 + x_2 + x_3 + x_4 + x_5 \\
c &= x_1 + x_2 + x_3 \\
d^\mu &= x_1 p_{34}^\mu + x_2 p_{134}^\mu + x_7 p_4^\mu \\
e^\mu &= x_1 p_{34}^\mu + x_2 p_{134}^\mu + x_5 p_3^\mu \\
f &= x_1 s.
\end{aligned}$$

With the change of variables

$$\begin{aligned} k^\mu &\to K^\mu - \frac{cL^\mu}{a} + \frac{ce^\mu - bd^\mu}{ab - c^2}, \\ l^\mu &\to L^\mu + \frac{cd^\mu - ae^\mu}{ab - c^2}, \end{aligned}$$

we can diagonalise the exponent

$$\sum_{i=1}^{n} x_i A_i = a\, K^2 + \frac{ab - c^2}{a}\, L^2 + \frac{\mathcal{Q}}{\mathcal{P}},$$

with

$$\begin{aligned} \mathcal{Q} &= a\, e^2 + b\, d^2 - f(ab - c^2) - 2\, c\, e \cdot d \\ \mathcal{P} &= ab - c^2, \end{aligned}$$

and we can integrate out the loop momenta $K$ and $L$ to obtain

$$\begin{aligned} I^D\left(\nu_1 \ldots \nu_n\right) &= \left( \prod_{i=1}^{n} \frac{(-1)^{\nu_i}}{\Gamma(\nu_i)} \int_0^\infty dx_i\, x_i^{\nu_i - 1} \right) \frac{1}{\mathcal{P}^{D/2}} \exp(\mathcal{Q}/\mathcal{P}) \\ &\equiv \int \mathcal{D}x \;\; \mathcal{I} \end{aligned}$$

# Tensor integral

With tensor integrals we perform the same shift of the momenta. For example

$$\int \frac{d^D k}{i\pi^{D/2}} \int \frac{d^D l}{i\pi^{D/2}} \frac{l^\mu}{A_1^{\nu_1} \ldots A_n^{\nu_n}} = \left( \prod_{i=1}^{n} \frac{(-1)^{\nu_i}}{\Gamma(\nu_i)} \int_0^\infty dx_i\, x_i^{\nu_i - 1} \right) \times \frac{1}{\mathcal{P}^{D/2}} \exp(\mathcal{Q}/\mathcal{P}) \frac{c d^\mu - a e^\mu}{\mathcal{P}}$$

where we have used

$$l^\mu \to L^\mu + \frac{c d^\mu - a e^\mu}{\mathcal{P}},$$

and the fact that the integral with $L^\mu$ vanishes (odd term).

Recalling that $a, b, \ldots e^\mu$ are linear function in $x_i$, we can absorb

- the extra factor of $x_i$

$$\frac{(-1)^{\nu_i} x_i^{\nu_i - 1}}{\Gamma(\nu_i)} x_i \implies -\nu_i \frac{(-1)^{\nu_i+1} x_i^{\nu_i}}{\Gamma(\nu_i + 1)} \equiv -\nu_i \mathbf{i}^+$$

$$\mathbf{i}^\pm I^D(\ldots, \nu_i, \ldots) = I^D(\ldots, \nu_i \pm 1, \ldots)$$

- the extra factor of $\mathcal{P}$

$$\frac{1}{\mathcal{P}^{D/2}} \frac{1}{\mathcal{P}} \implies \frac{1}{\mathcal{P}^{(D+2)/2}},$$

so that $1/\mathcal{P}$ acts as a dimension increaser

$$\frac{1}{\mathcal{P}} \implies \mathbf{d}^+, \qquad \mathcal{P} \implies \mathbf{d}^-,$$

where

$$\mathbf{d}^\pm I^D = I^{D\pm 2}.$$

Schematically

$$I^D\left(\ldots,\nu_i,\ldots,\nu_j,\ldots\right)[l^\mu] = \sum \nu_i\nu_j\, p_k^\mu\, I^{D+2}\left(\ldots,\nu_i+1,\ldots,\nu_j+1,\ldots\right)$$

This can be generalized for any tensor

$$\begin{aligned}
I^D[k^\mu k^\nu] &= \int \mathcal{D}x \left[\left(\frac{ce^\mu - bd^\mu}{\mathcal{P}}\right)\left(\frac{ce^\nu - bd^\nu}{\mathcal{P}}\right) - \frac{b}{2\mathcal{P}} g^{\mu\nu}\right] \mathcal{I} \\
I^D[k^\mu l^\nu] &= \int \mathcal{D}x \left[\left(\frac{ce^\mu - bd^\mu}{\mathcal{P}}\right)\left(\frac{cd^\nu - ae^\nu}{\mathcal{P}}\right) + \frac{c}{2\mathcal{P}} g^{\mu\nu}\right] \mathcal{I} \\
I^D[l^\mu l^\nu] &= \int \mathcal{D}x \left[\left(\frac{cd^\mu - ae^\mu}{\mathcal{P}}\right)\left(\frac{cd^\nu - ae^\nu}{\mathcal{P}}\right) - \frac{a}{2\mathcal{P}} g^{\mu\nu}\right] \mathcal{I}
\end{aligned}$$

and so on.

- Each extra factor of $a, b, \ldots e^\mu \Longrightarrow \nu_i \to \nu_i + 1$
- Each extra factor of $\mathcal{P} \Longrightarrow D \to D+2$

In this way, the task to compute tensor integrals is moved to the computation of scalar integrals with

1. higher powers of the propagators
2. in higher dimensions.

# 1. Higher powers of the propagators

## Recurrence relations

We do not need actually to compute all of the scalar integrals with different powers of the propagators because we can derive many identities that connect these integrals. Two ways to derive these identities:

1. Using the integration-by-parts method. We impose that the integral of the total derivative vanishes (no contribution coming from the surface)

$$\int \frac{d^D k}{i\pi^{D/2}} \int \frac{d^D l}{i\pi^{D/2}} \frac{\partial}{\partial a^\mu} \left[ b^\mu f(k,l,p_i) \right] = 0,$$

where, for the crossed box,

$$\begin{aligned} a^\mu &= k^\mu,\, l^\mu \\ b^\mu &= k^\mu,\, l^\mu,\, p_1^\mu,\, p_3^\mu,\, p_4^\mu \\ f(k,l,p_i) &= \frac{1}{A_1^{\nu_1} A_2^{\nu_2} A_3^{\nu_3} A_4^{\nu_4} A_5^{\nu_5} A_6^{\nu_6} A_7^{\nu_7}}. \end{aligned}$$

We can then derive 10 identities.

2) Exploiting the Lorentz invariance of the Feynman diagrams. In fact, since the Feynman integral is a function only of scalar products of the external momenta, it is invariant under the (infinitesimal) rotation

$$\int \frac{d^D k}{i\pi^{D/2}} \int \frac{d^D l}{i\pi^{D/2}}\, f(k,l,p_i) = \int \frac{d^D k}{i\pi^{D/2}} \int \frac{d^D l}{i\pi^{D/2}}\, f(k,l,p_i').$$

where

$$p_i'^{\mu} = \Lambda^{\mu}{}_{\nu}\, p_i^{\nu}, \qquad \Lambda_{\mu\nu} = g_{\mu\nu} + \delta\, \epsilon_{\mu\nu}, \qquad \epsilon_{\mu\nu} = -\epsilon_{\nu\mu}.$$

Expanding in a Taylor series in $\delta$ we obtain

$$\int \frac{d^D k}{i\pi^{D/2}} \int \frac{d^D l}{i\pi^{D/2}} \sum_{j=1}^{3} \frac{\partial f(k,l,p_i)}{\partial p_j^{\mu}}\, \epsilon^{\mu}{}_{\nu}\, p_j^{\nu} = 0.$$

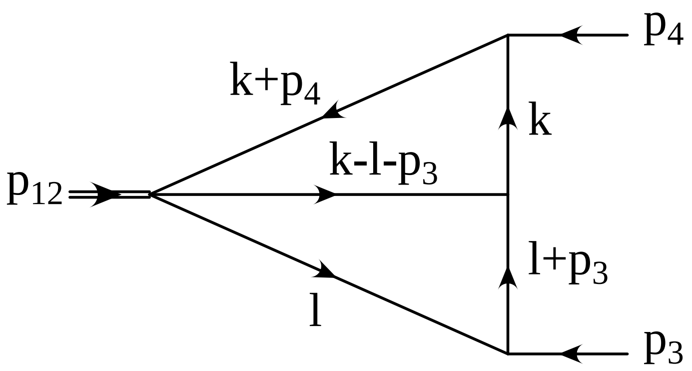
Using the three independent external momenta, we can build three independent antisymmetric tensors

$$\begin{aligned} \epsilon_1^{\mu\nu} &= p_1^{\mu}\, p_2^{\nu} - p_2^{\mu}\, p_1^{\nu}, \\ \epsilon_2^{\mu\nu} &= p_1^{\mu}\, p_3^{\nu} - p_3^{\mu}\, p_1^{\nu}, \\ \epsilon_3^{\mu\nu} &= p_2^{\mu}\, p_3^{\nu} - p_3^{\mu}\, p_2^{\nu}, \end{aligned}$$

that give rise to three more identities.

We have $10+3 = 13$ identities to exploit!!

# Example




$$
\begin{aligned}
A_1 &= (k+p_4)^2 & \qquad A_2 &= l^2 \\
A_3 &= (l+p_3)^2 & \qquad A_4 &= k^2 \\
A_5 &= (k-l-p_3)^2
\end{aligned}
$$

$$
\text{IBP} \Longrightarrow 0 = \int \frac{d^D k}{i\pi^{D/2}} \int \frac{d^D l}{i\pi^{D/2}} \, \frac{\partial}{\partial k^\mu} \left[ k^\mu \frac{1}{A_1^{\nu_1} A_2^{\nu_2} A_3^{\nu_3} A_4^{\nu_4} A_5^{\nu_5}} \right]
$$

$$
\begin{aligned}
0 \quad = \quad & \int \frac{d^D k}{i\pi^{D/2}} \int \frac{d^D l}{i\pi^{D/2}} \, \frac{1}{A_1^{\nu_1} A_2^{\nu_2} A_3^{\nu_3} A_4^{\nu_4} A_5^{\nu_5}} \\
& \times \left[ D - \nu_4 \frac{2\, k \cdot k}{A_4} - \nu_1 \frac{2\, k \cdot (k+p_4)}{A_1} - \nu_5 \frac{2\, k \cdot (k-l-p_3)}{A_5} \right]
\end{aligned}
$$

The scalar products can be rewritten as a combination of propagators

$$
\begin{aligned}
2\, k \cdot k &= 2\, A_4 \\
2\, k \cdot (k+p_4) &= A_1 + A_4 \\
2\, k \cdot (k-l-p_3) &= A_5 + A_4 - A_3
\end{aligned}
$$

We finally obtain

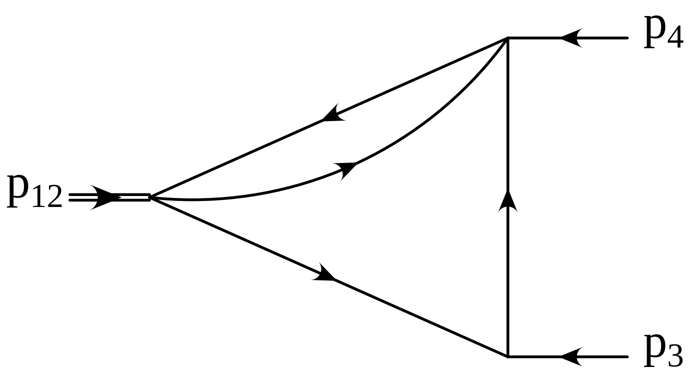
$$
\begin{aligned}
0 &= \int \frac{d^D k}{i\pi^{D/2}} \int \frac{d^D l}{i\pi^{D/2}} \frac{1}{A_1^{\nu_1} A_2^{\nu_2} A_3^{\nu_3} A_4^{\nu_4} A_5^{\nu_5}} \\
&\quad \times \left[ (D - 2\nu_4 - \nu_1 - \nu_5) - \nu_1 \frac{A_4}{A_1} - \nu_5 \frac{A_4}{A_5} + \nu_5 \frac{A_3}{A_5} \right]
\end{aligned}
$$

This identity allow us to build recurrence relations between integrals with fewer powers of $A_3$ and $A_4$ and higher powers of $A_5$ and $A_1$.

In a more compact form, using the raising and lowering operators

$$
(D - 2\nu_4 - \nu_1 - \nu_5) = \nu_1 \mathbf{1}^+ \mathbf{4}^- + \nu_5 \mathbf{5}^+ \left( \mathbf{4}^- - \mathbf{3}^- \right)
$$

When one propagator "disappear" from the integral, we obtain a simpler-topology diagram, that we call pinchings. For example, by repeated application of the previous recurrence relation, it may happen that the power of propagator 4, $\nu_4$, reduces to 0, and we obtain

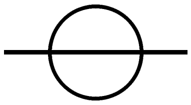
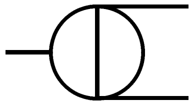



that is just a triangle with a bubble insertion, that is is a simple product of gamma functions.

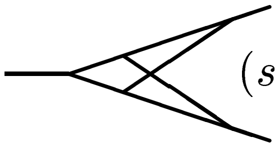
# Master integrals

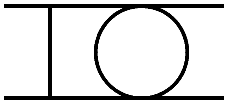
Using the 13 identities we can express ANY scalar crossed box with arbitrary powers of the propagators as a linear combination of the following integrals, that, therefore, are called master integrals [a]

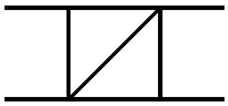
- trivial topologies:

$\mathrm{Sunset}(s) =$  $(s)$ $\qquad$ $\mathrm{Tri}(s) =$  $(s)$

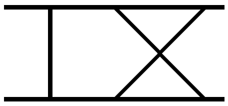
- less trivial topologies

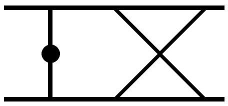
$\mathrm{Xtri}(s) =$  $(s)$

$\mathrm{Bbox}(s,t) =$  $(s,t)$

$\mathrm{Dbox}(s,t) =$  $(s,t)$

- non-trivial topologies

$\mathrm{Xbox}_1(s,t) =$  $(s,t)$

$\mathrm{Xbox}_2(s,t) =$  $(s,t)$

[a] C. Anastasiou, T. Gehrmann, E. Remiddi J.B. Tausk and C.O, *Nucl. Phys.* **B580** (2000) 577, `hep-ph/00032611`.

# 2. Dimensional shift

The second point to address is how to relate integral in $D+2$ dimensions with integral in $D$ dimensions.

Remember that

$$\mathcal{P} \Longleftrightarrow \mathbf{d}^- \qquad \text{and that} \qquad \mathcal{P} = ab - c^2$$

where $a$, $b$ and $c$ are linear in the $x_i$ $\Longrightarrow$

$$I^{(D-2)}\left(\ldots, \nu_i, \ldots, \nu_j, \ldots\right) = \sum \nu_i \nu_j I^{D}\left(\ldots, \nu_i + 1, \ldots, \nu_j + 1, \ldots\right)$$

But any scalar integral with arbitrary powers of the propagators can be expressed a sum of master integrals!

For example, we can obtain

$$\begin{aligned}
\mathrm{Xbox_1}^{D+2}(s,t) &= a_1 \mathrm{Xbox_1}^{D}(s,t) + a_2 \mathrm{Xbox_2}^{D}(s,t) + \text{pinchings} \\
\mathrm{Xbox_2}^{D+2}(s,t) &= a_1 \mathrm{Xbox_1}^{D}(s,t) + a_2 \mathrm{Xbox_2}^{D}(s,t) + \text{pinchings}
\end{aligned}$$

where

$$\begin{aligned}
a_1 &= \frac{6}{512} \frac{(D-4)s^2\left[(3D-14)(3D-10)s^2 - 4(5D^2-39D+74)tu\right]}{(D-3)^3(2D-9)(2D-7)tu} \\
a_2 &= \frac{(D-6)(D-4)s\left[4(D-3)tu - 3(3D-10)s^2\right]}{256(D-5)(D-3)^3(2D-9)(2D-7)} \\
b_1 &= \frac{s\left[3(D-4)(3D-14)s^2 - 4(D-6)(D-5)tu\right]}{64(D-3)(2D-9)(2D-7)tu} \\
b_2 &= -\frac{(D-6)\left[4(D-3)tu + 3(D-4)s^2\right]}{64(D-5)(D-3)(2D-9)(2D-7)}
\end{aligned}$$

## Analytic expansions of the masters integrals

Many people [a] contributed to compute the analytic expression in $D$, or at least the analytic expansion in $\epsilon$, where $D = 4 - 2\epsilon$ of the master integrals. The only missing integral was the second master crossed box Xbox$_2(s,t)$.

No need of any hard calculation! In fact, remembering

$$\mathrm{Xbox}^D(\nu_1 \ldots \nu_n) = \left( \prod_{i=1}^{n} \frac{(-1)^{\nu_i}}{\Gamma(\nu_i)} \int_0^\infty dx_i \, x_i^{\nu_i - 1} \right) \frac{1}{\mathcal{P}^{D/2}} \exp(\mathcal{Q}/\mathcal{P})$$

and that the only dependence from the physical scales is in $\mathcal{Q}$

$$\mathcal{Q} = x_2(x_5 x_6 - x_4 x_7) t + f(x_i) s.$$

We can differentiate both member with respect to $t$ and set $\nu_i$ to one, to obtain Xbox$_1$, and $\nu_i$ to one except $\nu_2 = 2$ to obtain Xbox$_2$

$$\begin{aligned}
\frac{\partial}{\partial t} \mathrm{Xbox}_1^D(s,t) &= \mathrm{Xbox}^{D+2}(1,2,1,2,1,1,2;s,t) \\
&\quad - \mathrm{Xbox}^{D+2}(1,2,1,1,2,2,1;s,t) \\
\frac{\partial}{\partial t} \mathrm{Xbox}_2^D(s,t) &= 2\,\mathrm{Xbox}^{D+2}(1,3,1,2,1,1,2;s,t) \\
&\quad - 2\,\mathrm{Xbox}^{D+2}(1,3,1,1,2,2,1;s,t)
\end{aligned}$$

and reduce the rhs with the help of the reduction formulae and of the dimensional shift.

[a] C.Anastasiou, E.E. Boos, A.I. Davydychev, E.W.N.Glover, R.J. Gonsalves, V.A.Smirnov, J.B.Tausk, O.L.Veretin . . .

We obtain

$$
\begin{aligned}
\frac{\partial}{\partial t}\mathrm{Xbox}_1^D(s,t) &= \frac{1}{t-u}\left[\frac{(D-4)s^2-4tu}{2tu}\,\mathrm{Xbox}_1^D(s,t)\right. \\
&\qquad \left. -\frac{(D-6)s}{2(D-5)}\,\mathrm{Xbox}_2^D(s,t)+\text{pinchings}\right] \\
\frac{\partial}{\partial t}\mathrm{Xbox}_2^D(s,t) &= \frac{1}{t-u}\left[\frac{2(D-5)^2 s}{tu}\,\mathrm{Xbox}_1^D(s,t)\right. \\
&\qquad \left. -\frac{(D-6)(u^2+t^2)}{tu}\,\mathrm{Xbox}_2^D(s,t)+\text{pinchings}\right]
\end{aligned}
$$

Since the analytic expansion for $\mathrm{Xbox}_1$ has been computed using Mellin-Barnes techniques [a]

- we can use the first equation to derive the analytic expansion for $\mathrm{Xbox}_2$
- we can use the second equation as an identity to check BOTH $\mathrm{Xbox}_1$ and $\mathrm{Xbox}_2$.

[a] J.B. Tausk, *Phys. Lett.* **B469** (1999) 225

## Second master integral $\mathrm{Xbox}_2$

$$\mathrm{Xbox}_2^D(s,t) = \Gamma^2(1+\epsilon)\left\{\frac{G_1(t,u)}{s^3 t} + \frac{G_2(t,u)}{s^2 t^2} + \frac{G_1(u,t)}{s^3 u} + \frac{G_2(u,t)}{s^2 u^2}\right\},$$

$$\begin{aligned}
G_1(t,u) = s^{-2\epsilon}\Bigg\{ & \frac{6}{\epsilon^3} + \frac{1}{\epsilon^2}(32 - 6T - 6U) \\
& + \frac{1}{\epsilon}\left(1 - 12\pi^2 - 24T + T^2 - 24U + 16TU + U^2\right) - 43 - 18T + 13T^2 + \frac{8}{3}T^3 \\
& - 18U + 16TU + 11T^2U + 13U^2 - 20TU^2 + \frac{8}{3}U^3 + \pi^2\left(17T + 17U - \frac{112}{3}\right) \\
& - 122\zeta(3) + 62T\,\mathrm{Li}_2\left(-\frac{t}{s}\right) - 62\,\mathrm{Li}_3\left(-\frac{t}{s}\right) + 62\,\mathrm{S}_{1,2}\left(-\frac{t}{s}\right) \\
& + i\pi\left[\frac{1}{\epsilon}(16 + 6T + 6U) - 34 - 9\pi^2 - 6T - 10T^2 - 6U + 14TU - 10U^2\right]\Bigg\},
\end{aligned}$$

$$\begin{aligned}
G_2(t,u) = s^{-2\epsilon}\Bigg\{ & -\frac{2}{\epsilon^4} + \frac{1}{\epsilon^3}\left(-8 + \frac{5}{2}T + \frac{7}{2}U\right) \\
& + \frac{1}{\epsilon^2}\left(-\frac{29}{2} - \frac{5}{12}\pi^2 + 7T - T^2 + 20U - 4TU - U^2\right) \\
& + \frac{1}{\epsilon}\Bigg[-\frac{1}{2} + 17T + 2T^2 - \frac{T^3}{3} + \frac{\pi^2}{6}(14 + 5T - 29U) + 13U - 28TU - 4U^2 \\
& \quad + 3TU^2 - U^3 + \frac{19}{2}\zeta(3) - 2T\,\mathrm{Li}_2\left(-\frac{t}{s}\right) + 2\,\mathrm{Li}_3\left(-\frac{t}{s}\right) - 2\,\mathrm{S}_{1,2}\left(-\frac{t}{s}\right)\Bigg] \\
& + \frac{37}{2} + \frac{37}{40}\pi^4 + 7T - 5T^2 - \frac{22}{3}T^3 + \frac{2}{3}T^4 + 5U - 20TU + \frac{8}{3}T^3U - 2U^2 \\
& + 24TU^2 - T^2U^2 - 8U^3 - \frac{4}{3}TU^3 + \frac{4}{3}U^4 \\
& + \frac{\pi^2}{6}\left(79 - 22T - 5T^2 - 200U + 76TU + 25U^2\right) + (68 - 13T - 33U)\,\zeta(3) \\
& + \left(10\pi^2 - 32T + 17T^2 + 12TU\right)\mathrm{Li}_2\left(-\frac{t}{s}\right) + (32 - 60T - 12U)\,\mathrm{Li}_3\left(-\frac{t}{s}\right) \\
& + (28T - 6U - 32)\,\mathrm{S}_{1,2}\left(-\frac{t}{s}\right) - 26\,\mathrm{S}_{1,3}\left(-\frac{t}{s}\right) - 36\,\mathrm{S}_{2,2}\left(-\frac{t}{s}\right) + 86\,\mathrm{Li}_4\left(-\frac{t}{s}\right) \\
& + i\pi\Bigg[\frac{2}{\epsilon^3} + \frac{1}{\epsilon^2}(11 - T + U) + \frac{1}{\epsilon}\left(1 - \frac{31}{6}\pi^2 - 10T - 2T^2 + 4U - 2TU - 2U^2\right) \\
& \quad + 11 + 4T - 2T^2 + \frac{10}{3}T^3 + \frac{\pi^2}{3}(-65 + 28T - U) + 2U - 8TU - 8U^2 \\
& \quad + 2U^3 - 89\zeta(3) + (14T + 18U)\,\mathrm{Li}_2\left(-\frac{t}{s}\right) - 32\,\mathrm{Li}_3\left(-\frac{t}{s}\right) + 44\,\mathrm{S}_{1,2}\left(-\frac{t}{s}\right)\Bigg]\Bigg\},
\end{aligned}$$

$$T = \log(-t/s) \qquad U = \log(-u/s).$$

# Conclusions and first results

All the ingredients for the computation of $2 \to 2$ scattering amplitudes at two-loops are there

- the tensor reduction of all the involved topologies is known
- the analytic expansions of all the master integrals are known too.

## Amplitudes for physical processes

- $e^+e^- \to \mu^+\mu^-$: 9 Oct 2000, by Bern, Dixon and Ghinculov, `hep-ph/0010075`.
- $q\overline{q} \to q'\overline{q}'$: to appear in a few hours/days by C. Anastasiou, E.W.N Glover, M.E. Tejeda-Yeomans, C.O.
- $q\overline{q} \to q\overline{q}$: we have already computed the expansions. We need time to compress the answer.
- $gg \to gg$: in the next future.