O'Mega:
An Optimizing Matrix Element Generator

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The number of tree Feynman diagrams w/ n legs in vanilla ϕ^3-theory is

$$F(n) = (2n-5)!! = (2n-5)\cdot(2n-7)\cdot\ldots\cdot 3\cdot 1$$

n	$F(n)$	$P(n)$
4	3	3
5	15	10
6	105	25
7	945	56
8	10395	119
9	135135	246
10	2027025	501
11	34459425	1012
12	654729075	2035

☹ computational costs grow beyond all reasonable limits

☹ gauge theory cancellations cause loss of precision

Number of independent momenta

$$P(n) = \frac{2^n-2}{2} - n = 2^{n-1} - n - 1$$

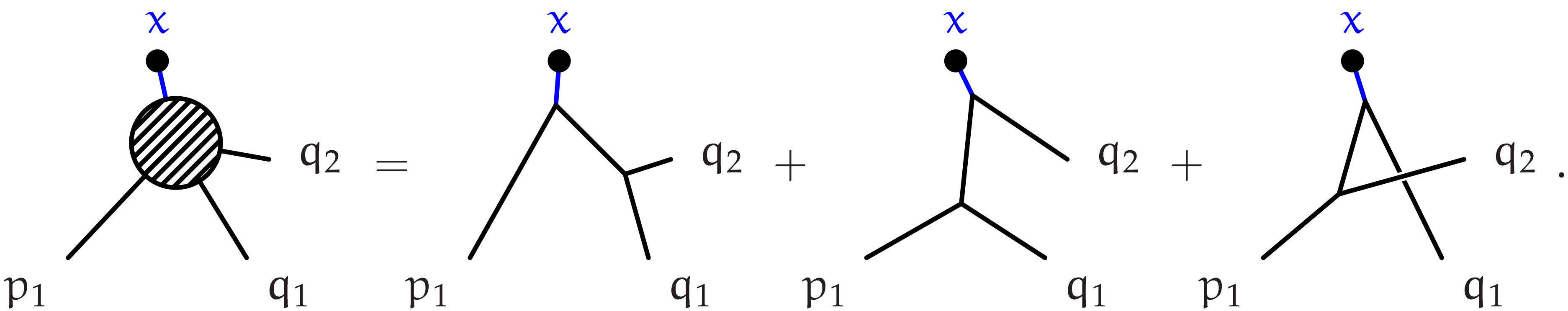
$\therefore$ Feynman diagrams extremely redundant for many particles in the final state!

☹ terms much too large to expect any help from common subexpression elimination by optimizing compilers that don't understand any physics!

One particle off-shell wave functions (1POWs) are obtained from Greensfunctions by applying the LSZ reduction formula to all but one line:

$$W(x;p_1,\ldots,p_n;q_1,\ldots,q_m) = \langle\phi(q_1),\ldots,\phi(q_m);\text{out}|\Phi(x)|\phi(p_1),\ldots,\phi(p_n);\text{in}\rangle\ .$$

E. g. $\langle\phi(q_1),\phi(q_2);\text{out}|\Phi(x)|\phi(p_1);\text{in}\rangle$ in unflavored scalar ϕ^3-theory at tree level



- the set of all 1POWs at tree level grows exponentially and can be constructed recursively from other 1POWs at tree level (the recursion terminates at the external wave functions).

(rhetorical) question: can we find keystones K with

$$T = \sum_{i=1}^{F(n)} D_i = \sum_{k,l,m=1}^{P(n)} K^3_{f_k f_l f_m}(p_k,p_l,p_m) W_{f_k}(p_k) W_{f_l}(p_l) W_{f_m}(p_m) + \text{quartic} + \ldots$$

There are two extreme solutions:

1. one maximally symmetric
2. n maximally asymmetric

The maximally asymmetric solutions are simple: pick any external line j and make the adjacent vertex a keystone. Equivalently, calculate the 1POW for particle j and put the amputated line on-shell:

$$T = W_{f_j}(p_j)\Big|_{\text{amp.,on-shell}}$$

- symbolic equivalent of the numerical Schwinger-Dyson equations of HELAC (cf. talk by Costas Papadopoulos)
- ☺ empirically, a correct implementation will require a few percent less multiplications in the amplitude compared to the symmetric solution below
- ☹ numerical problems have a few more steps to accumulate compared to the symmetric solution below
- ∴ more testing with sensitive gauge theory amplitudes required

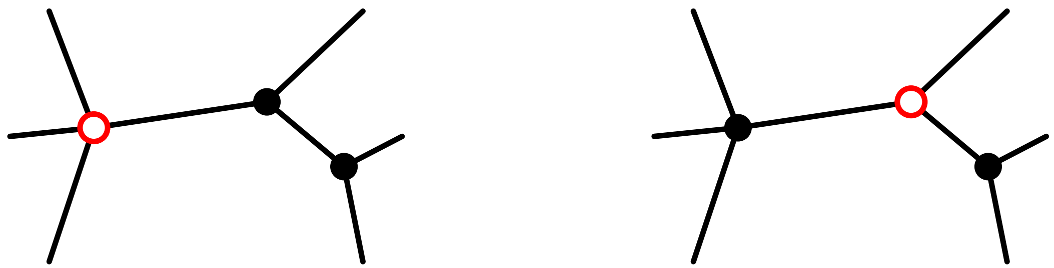
The maximally symmetric solutions (corresponding to the numerical approach of ALPHA) are built from balanced inequivalent partitions of external momenta:

n	$\sum$	$\sum$
4	4	$1 \cdot (1,1,1,1) + 3 \cdot (1,1,2)$
5	26	$1 \cdot (1,1,1,1,1) + 10 \cdot (1,1,1,2) + 15 \cdot (1,2,2)$
6	236	$1 \cdot (1,1,1,1,1,1) + 15 \cdot (1,1,1,1,2) + 40 \cdot (1,1,1,3)$
		$+ 45 \cdot (1,1,2,2) + 120 \cdot (1,2,3) + 15 \cdot (2,2,2)$

Subtlety: partitions for an even number of external lines of the type

$$(n_1,\ n_2,\ \ldots,\ n_{d-1},\ n_1 + n_2 + \cdots + \ldots + n_{d-1})$$

are two-fold degenerate, e. g. $(1,1,1,3)$ and $(1,2,3)$ contain the same diagram



$\therefore$ choose representatives consistently (with a lot of help from Mauro Moretti).

$F(d_{max}, n) =$ # of Feynman diagrams with n external legs

in unflavored

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 + \sum_{d=3}^{d_{max}} \frac{\lambda_d}{d!}\phi^d$$

theory. In a partition $N_{d,n} = \{n_1, n_2, \ldots, n_d\}$ with $n = n_1 + n_2 + \cdots + n_d$, there are

$$\tilde{F}(d_{max}, N_{d,n}) = \frac{1}{(1 + \delta_{n_d, n_1+n_2+\cdots+n_{d-1}})} \frac{n!}{|\mathcal{S}(N_{d,n})|} \prod_{i=1}^{d} \frac{F(d_{max}, n_i + 1)}{n_i!}$$

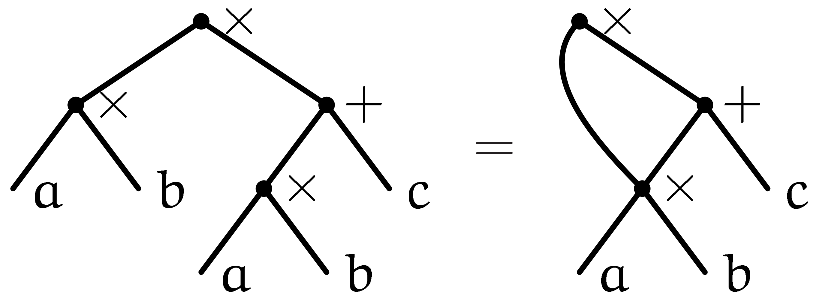
Feynman diagrams ($|\mathcal{S}(N)|$ the size of the symmetric group of N). Non trivial cross-check

$$F(d_{max}, n) = \sum_{d=3}^{d_{max}} \sum_{\substack{N=\{n_1,n_2,\ldots,n_d\} \\ n_1+n_2+\cdots+n_d=n \\ 1\le n_1\le n_2\le\cdots\le n_d\le\lfloor n/2\rfloor}} \tilde{F}(d_{max}, N)$$

☺ has been be checked numerically up to $n = \mathcal{O}(100)$.

Directed Acyclical Graphs (DAGs) are a more efficient representation for arithmetical expressions than the equivalent trees. E. g.:

$$ab(ab+c) =$$



- partial order *“depends on”* prohibits cycles
- any tree is equivalent to at least one DAG (itself)
- any DAG is equivalent to only one tree

Sets of trees and sets of DAGs can be characterized by the set of possible leaves and the possible branches

$\therefore$ there is a functor from the category of trees to the category of DAGs.

modern programming languages allow to make such algebraic relationships explicit and quickly lead to correct implementations.

By virtue of they recursive construction, tree level 1POWs form a DAG:

$\therefore$ find the smallest DAG that corresponds to a given tree (i. e. a sum of Feynman diagrams).

Systematic procedure:

Grow: starting from the external particles, build the tower of all 1POWs up to a given height (the height is always less than the number of external lines) and translate it to the equivalent DAG D.

Select: from D, determine all possible flavored keystones for the process under consideration and the 1POWs appearing in them.

Harvest: construct a sub-DAG $\mathrm{D}^* \subseteq \mathrm{D}$ consisting only of nodes that contribute to the 1POWs appearing in the flavored keystones.

Calculate: multiply the 1POWs as specified by the keystones and sum the keystones.

☺ the resulting expression contains no more redundancies!

Even for vector particles, the 1POWs are ‘almost’ physical objects and satisfy simple Ward Identities in unbroken gauge theories

$$\frac{\partial}{\partial x_\mu} \langle \text{out}|A_\mu(x)|\text{in}\rangle_{\text{amp.}} = 0$$

and in spontaneously gauge theories in R_ξ-gauge

$$\frac{\partial}{\partial x_\mu} \langle \text{out}|W_\mu(x)|\text{in}\rangle_{\text{amp.}} = \xi_W m_W \langle \text{out}|\phi_W(x)|\text{in}\rangle_{\text{amp.}} \,.$$

☺ code for matrix elements can optionally be instrumented to check these Ward identities.

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- Hindley-Milner type system
  - no holes (i. e. no casts): programs never crash
  - automatic type inference combines type safety with concise notation
  - parametric polymorphism facilitates code reuse by generic programming
- persistent data structures
  - all accessible values remain valid forever
  - ∴ manipulation of recursive data structures (trees, DAGs, etc.) straightforward, no user memory management required
- first class functions, e. g. composition

```
# let compose f g x = f (g x);;
val compose : ('a -> 'b) -> ('c -> 'a) -> 'c -> 'b
```

- functors mapping abstract data types (cf. categories), e. g.
  - one code creates a set from any ordered type (numbers, flavors, ...), another code creates a DAG from any forest (binary, n-ary, ...)

An ordered type:

```
module type Ord = sig type t val compare : t -> t -> int end
```

Forests are essentially parent/offspring relations:

```
module type Forest =
  sig
    module Nodes : Ord
    type node = Nodes.t
    type edge and children and t = edge * children
    val compare : t -> t -> int
    val for_all : (node -> bool) -> t -> bool
    val fold : (node -> 'a -> 'a) -> t -> 'a -> 'a
  end
```

The DAG functor creates the DAGs corresponding to a given forest:

```
module MakeDAG (F : Forest) :
    DAG with type node = F.node and type edge = F.edge
    and type children = F.children
```

DAGs are essentially condensed parent/offspring relations:

```
module type DAG =
  sig
    type node and edge and children and t
    val empty : t
    val add_node : node -> t -> t
    val add_offspring : node -> edge * children -> t -> t
    exception Cycle
    val is_node : node -> t -> bool
    val is_sterile : node -> t -> bool
    val is_offspring : node -> edge * children -> t -> bool
    val fold_nodes : (node -> 'a -> 'a) -> t -> 'a -> 'a
    val fold : (node -> edge * children -> 'a -> 'a) ->
      t -> 'a -> 'a
    val harvest : t -> node -> t -> t
    val lists : t -> (node * (edge * children) list) list
  end
```

`harvest` selects the minimal DAG, `list` yields a sequence of assignments.

- interactive toplevel and fast bytecode compiler for development and prototyping
- lean & mean implementation . . .

☺ optimizing compiler can reach 50% of the speed of raw C for the same algorithm

∴ can beat C and C++ with recursive algorithms

☺ O'Caml's transparent applicative functors more suitable for my applications than Standard-ML's generative functors

☺ available as free and open source software

☺ the bytecode variant works on all systems with an ANSI-C compiler, the native compiler is available for all relevant systems

☺ can be bootstrapped in $\mathcal{O}(10)$ minutes

$$\left\{\begin{array}{l}\left\{\begin{array}{l}\text{physics model}\\ \text{target language}\end{array}\right\}\\ \text{external particles}\end{array}\right\}\Longrightarrow\text{source code for amplitude}$$

- the physics model is currently specified in O'Caml, e. g.:

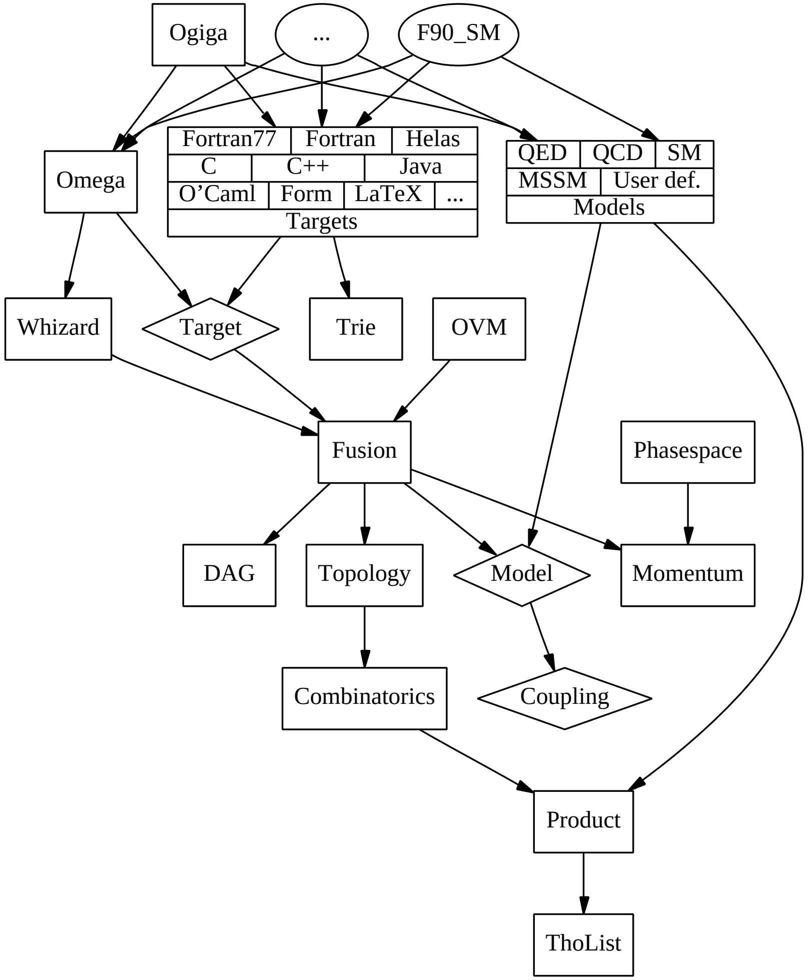
```
let charged_currents n =
  [ ((L (-n), Wm, N n), FBF (1, Psibar, VL, Psi), G_CC);
    ((N (-n), Wp, L n), FBF (1, Psibar, VL, Psi), G_CC);
    ((D (-n), Wm, U n), FBF (1, Psibar, VL, Psi), G_CC);
    ((U (-n), Wp, D n), FBF (1, Psibar, VL, Psi), G_CC) ]
```

  there will soon be a parser for CompHEP model files (probably also for GRACE) and later for O'Mega's own input language.

- the target language is specified as a O'Caml module. Inventing a specification language will not be economical in this case.

- the external particles are given on the command line (GUI later)

O'Caml's powerful module system supports a very flexible architecture w/ functors building applications from independent modules



The module `Targets` contains implementations of the signature `Target` for each target language & the module `Models` contains implementations of `Model` for each (class of) physics models.E. g. the application writing Fortran95 for the standard model is

```
module O = Omega.Make
             (Fusion.Binary)
             (Targets.Fortran)
             (Models.SM)
let _ = O.main ()
```

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Radiative corrections to four fermion production, standard model, unitarity gauge:

process $e^+e^- \to$	Diagrams #	Diagrams vertices	O'Mega #prop.	O'Mega vertices
$e^+\bar{\nu}_e d\bar{u}$	20	80	14	44
$e^+\bar{\nu}_e d\bar{u}\gamma$	146	730	36	151
$e^+\bar{\nu}_e d\bar{u}\gamma\gamma$	1112	6672	94	468
$e^+\bar{\nu}_e d\bar{u}\gamma\gamma\gamma$	12420	86940	168	1246
$e^+\bar{\nu}_e d\bar{u}\gamma\gamma\gamma\gamma$	138816	1110528	344	3746

- O'Mega amplitudes for up to 7 particles ("$2 \to 5$") tested against MADGRAPH

☺ agreement for random momenta always better than 10^{-11}

First realistic application

- simulation of six fermion final states in W^+W^- scattering for the TESLA Technical Design Report, using WHIZARD by Wolfgang Kilian as unweighted event generator.

Get it from `http://www.ikp.physik.tu-darmstadt.de/~ohl/omega/`.

- QCD
  - up to two colored particles are already handled
  - factorization for many-jet final states?
- Supersymmetry and MSSM

  ☺ Jürgen Reuter (Darmstadt) has added unified support for Dirac and Majorana fermions using the Feynman rules of Ansgar Denner et al.
- O'Mega Virtual Machine

  $\because$ most time is spent in non-trivial vertex evaluations for vectors and spinors, that take $\mathrm{O}(10)$ complex multiplications

  $\therefore$ virtual vertex evaluation machines can challenge native code and avoid compilations
- O'Giga: O'Mega Graphical Interface for Generation and Analysis
- O'Tera: O'Mega Tool for Evaluating Renormalized Amplitudes