# Optimization of symbolic evaluation of helicity amplitudes

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**Abstract.** We present a method for symbolic evaluation of Feynman amplitudes. We construct special polarization basis for spinor particles which produces compact expressions for tensor products of basis spinors.

Quantum amplitudes corresponded to Feynman diagrams depend on momenta of particles and their polarizations. Polarizations of bosons are described by some Lorentz vectors. Thus, in case, when only boson particles are involved, the amplitudes may be presented via dot-products and Levi-Civita tensors. However fermion polarizations are described by vectors in the spinor space, so the amplitude is expressed as a product of Dirac $\gamma$-matrices ended by two spinors (hereafter *fermion string*). Such an expression looks like *not complete evaluated* — on the step of numerical evaluation one has to perform summations over spinor and Lorentz indices. During the last few years effective algorithms for these summations were developed, e.g. [1, 2, 3, 4, 5, 6], and corresponding programs were created for automatic computation of Feynman amplitudes, e.g. [7, 8, 9, 10].

Further progress one can get, we believe, if indices summation is performed before numerical calculations. We refer on our experience with the CompHEP package [11], where the squared diagram technique is used. For processes with four particles in final state the number of squared diagrams becomes very large comparing with the amplitude, and corresponding symbolic answers are cumbersome. Nevertheless, CompHEP works effectively because it prepares analytical expression for each squared diagram in terms of dot-products, and, then, factorizes them. Also, analytical answers for diagrams can be used to resolve explicitly the cancellations between diagrams, important for gauge theories.

The idea of the fermion string symbolic evaluation is known long time ago [12]. Although a spinor can not be expressed in terms of Lorentz vectors, such a possibility exists for pairs of spinors:

$$w \otimes \bar{w}' \equiv w^i(p_1,\lambda_1)\bar{w}'_j(p_2,\lambda_2)\,, \tag{1}$$

where $p_1$ and $p_2$ are particle momenta, $\lambda_1$ and $\lambda_2$ are their polarizations. This matrix can be expressed through the Dirac $\gamma$-matrices convoluted with vectors $p_i$, $\lambda_i$ and some auxiliary vectors. Thus, the contribution of a fermion string, $\Gamma$, to the Feynman amplitude can be rewritten through the trace:

$$\bar{w}(p_2,\lambda_2)\Gamma\, w'(p_1,\lambda_1) = tr(\Gamma\, w \otimes \bar{w}')\,, \tag{2}$$

where this string, $\Gamma$, is some product of $\gamma$-matrices.

There are different approaches to the construction of tensor products like (1), see e.g. [13, 14, 15]. Here we propose a new method with the following advantages: 1) it produces most compact symbolic expressions for (1); 2) it gives formulae uniformly applicable in massive and massless cases, for Dirac and Majorana fermions; 3) it may be applied to interactions with the fermion number violation; 4) all phases of fermion states are fixed completely, so one can choose fermion pairs by different ways.

Spinors, corresponding to fermion external legs of the diagram, can be one of the following variants, $w \in \{u, v, u^c, v^c\}$ and $\bar{w}' \in \{\bar{u}, \bar{v}, \overline{u^c}, \overline{v^c}\}$. Here $(^c)$ denotes the operation of C-conjugation, the corresponding fermion strings appear, for example, in the case of interaction with fermion number violation [16]. Spinors $u$ and $v$ are defined as

$$\begin{aligned} <0|\psi(x)|\Omega_+(p,\lambda)> &= e^{-ipx}u(p,\lambda)\,, & (3)\\ <0|\bar{\psi}(x)|\Omega_-(p,\lambda)> &= e^{-ipx}\bar{v}(p,\lambda)\,, & (4) \end{aligned}$$

where $|0>$ is vacuum state, $\Omega_+$ ($\Omega_-$) is one-particle fermion (anti-fermion) state. The fermion states depend on the choice of the polarization axes and their complex phases are not fixed.

The characteristic property of spinors $u$ and $v$ is the *completeness*:

$$\begin{aligned} \Sigma_\lambda\, u(p,\lambda) \otimes \bar{u}(p,\lambda) &= \hat{p} + m & (5)\\ \Sigma_\lambda\, v(p,\lambda) \otimes \bar{v}(p,\lambda) &= \hat{p} - m & (6) \end{aligned}$$

where $m = \sqrt{p^2}$. One can say that any spinors, $u(p,\lambda)$ and $v(p,\lambda)$, satisfying these relations correspond to some polarization states $\Omega_+$ and $\Omega_-$.

By definition, for Majorana fermion one has $\psi(x) = \psi(x)^c$ and $\Omega_+(p,\lambda) = \Omega_-(p,\lambda)$. If one substitutes these identities into (3,4) one gets

$$u(p,\lambda) = C\bar{v}(p,\lambda)^T = v(p,\lambda)^c \,. \tag{7}$$

Thus, we conclude that spinor basis should satisfy relation (7) in order to construct an algorithm for evaluation of fermion strings with Majorana fermions. Hereafter we assume that spinors $u$ and $v$ satisfy this relation.

Following to [1, 2] we construct $u$ and $v$ spinors in arbitrary point of the momentum space as a projection of spinors defined at some auxiliary point $p_0$:

$$\begin{aligned} u(p,\lambda) &= N(m+\hat{p})u(p_0,\lambda) \,, & (8)\\ v(p,\lambda) &= N(m-\hat{p})v(p_0,\lambda) \,, & (9) \end{aligned}$$

where $N = 1/\sqrt{2[(p_0 p)+m_0 m]}$. It is easy to check that this procedure agrees with the completeness (5, 6) at any point $p$ if it is satisfied at the point $p_0$. Also, it is enough to satisfy (7) at $p_0$ to get this relation at any point.

Let $p_0$ be a massless vector, in this case expressions appearing are simplified. In particular, equations (5) and (6) become identical, and, thus, (massless) $u$ and $v$ spinors are basis vectors in the same spinor subspace. It is well known, that left-handed and right-handed spinors, $\xi_- = -\gamma_5\xi_-$ and $\xi_+ = \gamma_5\xi_+$, may be chosen as elements of this basis, and completeness reads as $\xi_-\otimes\bar{\xi}_- + \xi_+\otimes\bar{\xi}_+ = \hat{p}_0$. Note that chiral spinors, $\xi_\pm$, are defined by these relations up to phases. Let us fix these phases.

At first, let us connect chiral spinors with massless $v$ spinor of different polarizations taken at the point $p_0$:

$$\xi_+ = v(p_0,+1) \,, \qquad \xi_- = v(p_0,-1) \,.$$

Then, from (7) one gets

$$u(p_0,+1) = \xi_+^c \,, \qquad u(p_0,-1) = \xi_-^c \,.$$

Note, then, that C-conjugation transforms the subspace of $u$-spinors to the subspace of $v$-spinors. For massless $p_0$ these subspaces are identical and, thus, invariant under the C-conjugation. From other side C-conjugation transforms left-handed spinor to right-handed spinor. So, sum of the phases of $\xi_\pm$ can be fixed by the relation

$$\xi_- = \xi_+^c \,. \tag{10}$$

As a result of our construction the following expressions for $u$ and $v$ spinors can be derived:

$$u(p,\pm 1) = v(p,\pm 1)^c = N(m+\hat{p})\xi_\mp \,, \tag{11}$$

$$v(p,\pm 1) = u(p,\pm 1)^c = N(m-\hat{p})\xi_\pm \,, \tag{12}$$

where $N = 1/\sqrt{2(p_0 p)}$.

By means of Dirac conjugation one immediately gets

$$\bar{u}(p,\pm 1) = \overline{v(p,\pm 1)^c} = N\bar{\xi}_\mp(m+\hat{p}) \,, \tag{13}$$

$$\bar{v}(p,\pm 1) = \overline{u(p,\pm 1)^c} = N\bar{\xi}_\pm(m-\hat{p}) \,. \tag{14}$$

Expressions (11 - 14) reduce the problem of evaluation of matrices $w\otimes\bar{w}'$ to the evaluation of $\xi_+\otimes\bar{\xi}_+$, $\xi_-\otimes\bar{\xi}_-$, $\xi_+\otimes\bar{\xi}_-$ and $\xi_-\otimes\bar{\xi}_+$. The first two products can be derived directly from the definition of chiral spinors:

$$\begin{aligned} \xi_+\otimes\bar{\xi}_+ &= \frac{1+\gamma_5}{2}\hat{p}_0 \,, & (15)\\ \xi_-\otimes\bar{\xi}_- &= \frac{1-\gamma_5}{2}\hat{p}_0 \,. & (16) \end{aligned}$$

However, expressions for last two products, $\xi_+\otimes\bar{\xi}_-$ and $\xi_-\otimes\bar{\xi}_+$, can not be derived yet, because they are sensitive to the multiplication of $\xi_\pm$ by opposite phase factors. Note that rotation of 3-dimension coordinate system around the space component of $p_0$, will produce the multiplication of $\xi_\pm$ by opposite phase factors. Thus, one needs to introduce additional auxiliary vectors to fix the reference frame, and (as a result) to fix phases, and (finally) to get a representation for $\xi_+\otimes\bar{\xi}_-$ and $\xi_-\otimes\bar{\xi}_+$. Let $\eta_1$ and $\eta_2$ are space-like vectors orthogonal to $p_0$:

$$\eta_1{\cdot}\eta_2 = 0 \,, \quad \eta_i{\cdot}p_0 = 0 \,, \quad \eta_i{\cdot}\eta_i = -1 \,. \tag{17}$$

Three vectors $(p_0, \eta_1, \eta_2)$ may be completed to the basis in Minkowsky space [1]. Now we need that this basis is right-handed oriented. It can be done by fixing the sign of the convolution of Levi-Civita tensor with basis vectors. We fix the basis orientation by the relation

$$\hat{p}_0 = i\gamma_5\hat{\eta}_1\hat{\eta}_2\hat{p}_0 \,. \tag{18}$$

If one constructs complex vectors $\eta \equiv (\eta_1 + i\eta_2)/2$ and $\eta^* \equiv (\eta_1 - i\eta_2)/2$, then (18) can be rewritten as

$$\hat{\eta}^*\hat{\eta}\hat{p}_0 = -(1/2)(1-\gamma_5)\hat{p}_0 \,. \tag{19}$$

Using this equation one can check that

$$\xi_- = \hat{\eta}^*\xi_+ \,. \tag{20}$$

Note that (16) defines $\xi_-$ uniquely up to a phase. Thus (20) fixes the difference of phases between $\xi_-$ and $\xi_+$ (whereas (10) fixes their sum). Now one can derive from (20) the second pair of products:

$$\xi_-\otimes\bar{\xi}_+ = \frac{1-\gamma_5}{2}\hat{\eta}^*\hat{p}_0 \,, \quad \xi_+\otimes\bar{\xi}_- = -\frac{1+\gamma_5}{2}\hat{\eta}\hat{p}_0 \,.$$

Note that these relations are some variant of the Bouchiat-Michel equations [17, 18].

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[1] say, by vector $q_0$, such that $q_0{\cdot}p_0 = 1$ and $q_0{\cdot}\eta_i = 0$.

Now we can write down the complete set of expressions for tensor products $w \otimes \bar{w}'$:

$$u(p,\lambda) \otimes \bar{u}(p',\lambda') = \tag{21}$$

$$= N(m+\hat{p}) \begin{pmatrix} 1-\gamma_5 & (1-\gamma_5)\hat{\eta}^* \\ -(1+\gamma_5)\hat{\eta} & 1+\gamma_5 \end{pmatrix} \hat{p}_0(m'+\hat{p}')\,;$$

$$u(p,\lambda) \otimes \bar{v}(p',\lambda') = \tag{22}$$

$$= N(m+\hat{p}) \begin{pmatrix} (1-\gamma_5)\hat{\eta}^* & 1-\gamma_5 \\ 1+\gamma_5 & -(1+\gamma_5)\hat{\eta} \end{pmatrix} \hat{p}_0(m'-\hat{p}')\,;$$

$$v(p,\lambda) \otimes \bar{u}(p',\lambda') = \tag{23}$$

$$= N(m-\hat{p}) \begin{pmatrix} -(1+\gamma_5)\hat{\eta} & 1+\gamma_5 \\ 1-\gamma_5 & (1-\gamma_5)\hat{\eta}^* \end{pmatrix} \hat{p}_0(m'+\hat{p}')\,;$$

$$v(p,\lambda) \otimes \bar{v}(p',\lambda') = \tag{24}$$

$$= N(m-\hat{p}) \begin{pmatrix} 1+\gamma_5 & -(1+\gamma_5)\hat{\eta} \\ (1-\gamma_5)\hat{\eta}^* & 1-\gamma_5 \end{pmatrix} \hat{p}_0(m'-\hat{p}')\,.$$

Here $N = 1/(4\sqrt{(pp_0)(p'p_0)})$, $m = \sqrt{p^2}$, $m' = \sqrt{p'^2}$, and dependence on the polarizations follows the rule: $\begin{pmatrix} ++ & +- \\ -+ & -- \end{pmatrix}$. The products $w \otimes \bar{w}'$ with C-conjugation may be derived now by means of relation (7).

Note, that for $m \neq 0$ the spinors $u$ and $v$ have spin directed along the axis $n = p/m - p_0(m/(p.p_0))$. In massless case spinors $u(p,+1)$ and $v(p,-1)$ are right-handed, whereas $u(p,-1)$ and $v(p,+1)$ are left-handed.

At this step one can conclude that in the proposed method the same set of auxiliary vectors ($p_0$, $\eta$, $\eta^*$) is introduced for arbitrary number of fermion strings in the amplitudes. Moreover, only one set can be used for all diagrams to fix relative phases of spinors for interference diagrams. One can compare with other methods where number of sets of auxiliary vectors depend on the number of fermion strings (see, e.g., corresponding discussion in [13]).

One can find further optimizations in the proposed technique. Indeed, vectors $\eta$ and $\eta^*$ in (21-24) are accompanied always by the $p_0$ vector. Let us define antisymmetric tensor $G_{\mu\nu} \equiv -i\varepsilon_{\mu\nu\alpha\beta}\eta^\alpha p_0^\beta$. Then one can check that $\hat{\eta}\hat{p}_0 = \frac{1}{2}G_{\mu\nu}\gamma^\mu\gamma^\nu$ and $\hat{\eta}^*\hat{p}_0 = \frac{1}{2}G^*_{\mu\nu}\gamma^\mu\gamma^\nu$. By substituting these relations in (2) one can hide the dependence on $\eta$, $\eta^*$ inside the $G$ tensor.

The $G$ tensor has some properties allowing one to perform the indices convolutions:

$$G_{\mu\nu}{p_0}^\mu = 0\,, \quad G_{\mu\nu}{G_{\mu'}}^\nu = 0\,, \quad {G_\mu}^\nu G^*_{\nu\mu'} = 2p_{0\mu}p_{0\mu'}\,,$$

$$g^{\mu\mu'}G_{\mu\nu}\varepsilon_{\mu'\alpha\beta\gamma} = 4i(g_{\nu\alpha}G_{\beta\gamma} - g_{\nu\beta}G_{\alpha\gamma} + g_{\nu\gamma}G_{\alpha\beta})\,.$$

As a result, one can get symbolic answer for the amplitude expressed via scalar products of particle momenta and auxiliary vector $p_0$, their convolutions with the Levi-Civita tensor, and particle momenta convolved with the $G$ tensor. The use of $G$ tensor optimizes the final expressions — if we expand $G$ tensor via scalar products two terms appear.

Then, one can expect that the amplitude contains both $G$ and $G^*$ tensors. However, the product of these tensors, $G(q_1,p_1)G^*(q_2,p_2)$, can be expressed via the dot-products and Levi-Civita tensors. So, one can connect fermion legs of the diagram in such a way that amplitudes will be presented via either $G$ or $G^*$ tensors, but never both together.

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