The differential equation method:
evaluation of complicated Feynman integrals

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Abstract

We discuss a progress in calculation of Feynman integrals which has been done with help of the differential equation method and demonstrate the results for a class of two-point two-loop diagrams.

The idea of the differential equation method (DEM) (see [1,2]): to apply the integration by parts procedure [3] to an internal n-point subgraph of a complicated Feynman diagram and later to represent new complicated diagrams, obtained here, as derivatives in respect of corresponding mass of the initial diagram.

Thus, we have got the differential equation for the initial diagram. The inhomogeneous term contains only more simpler diagrams. These simpler diagrams have more trivial topological structure and/or less number of loops [1] and/or ends [2].

Applying the procedure several times, we will able to represent complicated Feynman integrals (FI) and their derivatives (in respect of internal masses) through a set of quite simple well-known diagrams. Then, the results for the complicated FI can be obtained by integration several times of the known results for corresponding simple diagrams.

Sometimes it is useful (see [4]) to use external momenta (or some their functions) but not masses as parameters of integration.

The recent progress in calculation of Feynman integrals with help of the DEM.

1. The articles [5] and [6]:

a) The set of two-point two-loop FI with one- and two-mass thresholds has been evaluated by DEM (see Fig.1). The results are given on pages 3 and 4 and of some ot them have been known before (see disscussions in [5]). The check of the results has been done by Veretin program.

b) The set of three-point two-loop FI with one- and two-mass thresholds has been evaluated (the results of some ot them has been known before (see [6])) by a combination of DEM and Veretin program for calculation of first terms of FI small-moment expansion.

2. The article [7]:

The full set of two-point two-loop onshell master diagrams has been evaluated by DEM. The check of the results has been done by Kalmykov program.

3. The articles [8]:

The set of three-point and four-point two-loop massless FI has been evaluated.

Here we demonstrate the results of FI are displayed on Fig.1.

We introduce the notation for polylogarithmic functions:

$$\mathrm{Li}_a(z) = S_{a-1,1}(z), \quad S_{a+1,b}(z) = \frac{(-1)^{a+b}}{a!\,b!} \int_0^1 \frac{\log^a(t)\log^b(1-zt)}{t}\,dt. \tag{1}$$

We introduce also the following two variables

$$z = \frac{q^2}{m^2}, \quad y = \frac{1-\sqrt{z/(z-4)}}{1+\sqrt{z/(z-4)}}. \tag{2}$$

Then

$$\begin{aligned} q^2 \cdot I_1 = & -\frac{1}{2}\log^2(-z)\log(1-z) - 2\log(-z)\mathrm{Li}_2(z) + 3\mathrm{Li}_3(z) - 6S_{1,2}(z) \\ & - \log(1-z)\Big(\zeta_2 + 2\mathrm{Li}_2(z)\Big), \end{aligned} \tag{3}$$

$$\begin{aligned} q^2 \cdot I_5 = & \, 2\zeta_2\log(1+z) + 2\log(-z)\mathrm{Li}_2(-z) + \log^2(-z)\log(1+z) + 4\log(1+z)\mathrm{Li}_2(z) \\ & - 2\mathrm{Li}_3(-z) - 2\mathrm{Li}_3(z) + 2S_{1,2}(z^2) - 4S_{1,2}(z) - 4S_{1,2}(-z), \end{aligned} \tag{4}$$

$$\begin{aligned} q^2 \cdot I_{12} = & \, \mathrm{Li}_3(z) - 6\zeta_3 - \zeta_2\log y - \frac{1}{6}\log^3 y - 4\log y\,\mathrm{Li}_2(y) \\ & + 4\mathrm{Li}_3(y) - 3\mathrm{Li}_3(-y) + \frac{1}{3}\mathrm{Li}_3(-y^3), \end{aligned} \tag{5}$$

$$q^2 \cdot I_{13} = -6S_{1,2}(z) - 2\log(1-z)\Big(\zeta_2 + \mathrm{Li}_2(z)\Big), \tag{6}$$

$$
\begin{aligned}
q^2 \cdot I_{14} = & \log(2-z)\Big(\log^2(1-z) - 2\log(-z)\log(1-z) - 2\mathrm{Li}_2(z)\Big) \\
& - \frac{2}{3}\log^3(1-z) + \log(-z)\log^2(1-z) - 2\zeta_2\log(1-z) \\
& - S_{1,2}(1/(1-z)^2) + 2S_{1,2}(1/(1-z)) + 2S_{1,2}(-1/(1-z)) \\
& + \frac{1}{3}\log^3 y + \log^2 y\,\Big(2\log(1+y^2) - 3\log(1-y+y^2)\Big) \\
& - 6\zeta_3 - \mathrm{Li}_3(-y^2) + \frac{2}{3}\mathrm{Li}_3(-y^3) - 6\mathrm{Li}_3(-y) \\
& + 2\log y\,\Big(\mathrm{Li}_2(-y^2) - \mathrm{Li}_2(-y^3) + 3\mathrm{Li}_2(-y)\Big)\,, \qquad (7)
\end{aligned}
$$

$$
\begin{aligned}
q^2 \cdot I_{15} = & 2\mathrm{Li}_3(z) - \log(-z)\,\mathrm{Li}_2(z) + \zeta_2\log(1-z) \\
& + \frac{1}{6}\log^3 y - \frac{1}{2}\log^2 y\,\Big(8\log(1-y) - 3\log(1-y+y^2)\Big) \\
& - 6\zeta_3 - \frac{1}{3}\mathrm{Li}_3(-y^3) + 3\mathrm{Li}_3(-y) + 8\mathrm{Li}_3(y) \\
& + \log y\,\Big(\mathrm{Li}_2(-y^3) - 3\mathrm{Li}_2(-y) - 8\mathrm{Li}_2(y)\Big)\,, \qquad (8)
\end{aligned}
$$

$$
\begin{aligned}
q^2 \cdot I_{123} = & -\zeta_2\Big(\log(1-z) + \log y\Big) - 6\zeta_3 - \frac{3}{2}\log(1-y+y^2)\,\log^2 y \\
& + \mathrm{Li}_3(-y^3) - 9\mathrm{Li}_3(-y) - 2\log y\,\Big(\mathrm{Li}_2(-y^3) - 3\mathrm{Li}_2(-y)\Big)\,, \qquad (9)
\end{aligned}
$$

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q^2 \cdot I_{125} = -2\log^2 y\,\log(1-y) - 6\zeta_3 + 6\mathrm{Li}_3(y) - 6\log y\,\mathrm{Li}_2(y)\,, \qquad (10)
$$

$$
\begin{aligned}
q^2 \cdot I_{1234} = & -6\zeta_3 - 12\mathrm{Li}_3(y) - 24\mathrm{Li}_3(-y) \\
& + 8\log y\,\Big(\mathrm{Li}_2(y) + 2\mathrm{Li}_2(-y)\Big) + 2\log^2 y\,\Big(\log(1-y) + 2\log(1+y)\Big)\,. \qquad (11)
\end{aligned}
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